STEP 2005, Paper 1, Q1 Solution (4 pages; 11/2/26)

1 47231 is a five-digit number whose digits sum to $4+7+2+3+1=17$.

(i) Show that there are 15 five-digit numbers whose digits sum to 43. You should explain your reasoning clearly.

(ii) How many five-digit numbers are there whose digits sum to 39?

(i) **Method 1**

A 'case by case' approach can be adopted here. We could, for example, consider the different possibilities for the 1st digit:

79999: 1 rearrangement of 9999

89998: 4 rearrangements of 9998

99997: 4 rearrangements of 9997

99988: ${}^4C_2 = 6$ rearrangements of 9988

giving a total of 15

Method 2

Alternatively, we could categorise the numbers according to the number of 9s:

4 9s: 99997: 5 rearrangements

3 9s: 99988: $\frac{5!}{3!2!} = 10$ rearrangements

again, giving a total of 15

Method 3

There is also a more advanced approach that students wouldn't be expected to know. It could be used in the exam, but would need to be explained well. It is known as the 'stars and bars' method.

To find the number of ways of selecting n items from r objects (where n can be less than, equal to, or greater than r), when objects can be selected more than once (and the order of selection is not considered):

eg to choose 6 fruits from Oranges, Apples & Pears; one possible selection would be OOAPPP (not necessarily selected in that order)

The notation used in this example would be OO|A|PPP, or

$** \mid * \mid ***$ (assuming that the fruits were being considered in the order OAP)

Then *||***** would represent OPPPPP.

The total number of possible selections is the number of ways of placing the bars amongst the total number of places (8 in this example).

In general this is $\binom{n+(r-1)}{r-1}$

In (i), starting with 99999, we need to choose 2 [45-43] digits to subtract 1 from (possibly the same digit), in order to produce 43.

So $n = 2$ & $r = 5$, and the number of ways is $\binom{2+(5-1)}{5-1} = \binom{6}{4}$

$= \binom{6}{2} = \frac{6\times5}{2!} = 15$

(ii) The 2nd approach in (i) seems to be shorter than the 1st:

4 9s: 99993: 5 rearrangements

3 9s: 99984/99975/99966:

$\frac{5!}{3!} + \frac{5!}{3!} + \frac{5!}{3!2!} = 20 + 20 + 10 = 50$ rearrangements

2 9s: 99885/99876/99777 (we can consider the number of 8s here)

$\frac{5!}{2!2!} + \frac{5!}{2!} + \frac{5!}{2!3!} = 30 + 60 + 10 = 100$ rearrangements

1 9: 98886/98877 (again, considering the number of 8s)

$\frac{5!}{3!} + \frac{5!}{2!2!} = 20 + 30 = 50$ rearrangements

0 9s: 88887: 5 rearrangements

giving a total of $5 + 50 + 100 + 50 + 5 = 210$ rearrangements

Using the Stars & Bars method

$n = 45 - 39 = 6$ & $r = 5$, and the number of ways is

$$\binom{6 + (5 - 1)}{5 - 1} = \binom{10}{4} = \frac{10\times9\times8\times7}{4!} = 10 \times 3 \times 7 = 210$$