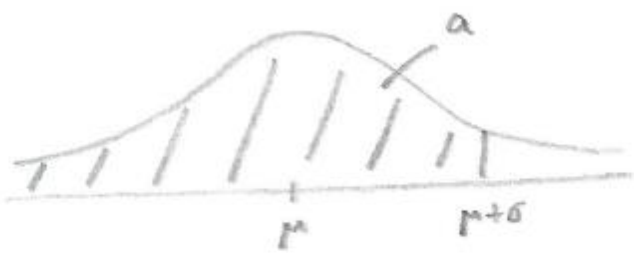
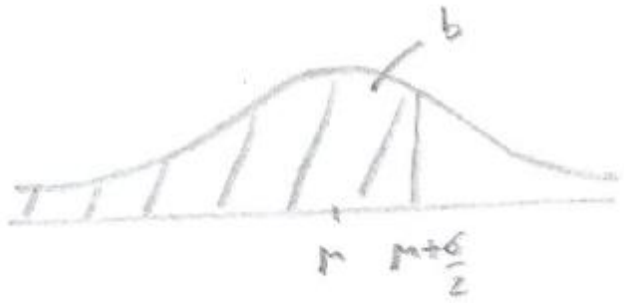
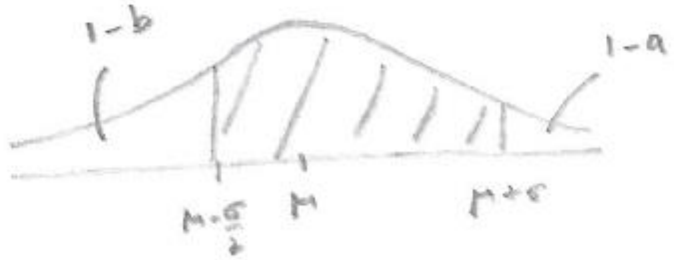
STEP 2005, Paper 1, Q13 - Solution (3 pages; 10/5/18)

[You can never be sure with this type of question whether the earlier parts are intended to be used later, or are just there as a warm-up exercise (as in this case, it seems).]

(a)







$P\left(\mu - \frac{\sigma}{2} \leq X \leq \mu + \sigma\right) = 1 - (1-b) - (1-a) = a + b - 1$

(b) $P\left(X \leq \mu + \frac{\sigma}{2} \middle| X \geq \mu - \frac{\sigma}{2}\right) = \frac{P(\mu - \frac{\sigma}{2} \leq X \leq \mu + \frac{\sigma}{2})}{P(X \geq \mu - \frac{\sigma}{2})}$

$= \frac{1-2(1-b)}{b} = \frac{2b-1}{b}$

(b)(i) [Some parts of this question involve $\leq$ etc; others $<$; obviously it makes no difference.]

Let Y be the volume for the unknown type of milk.

Let SM denote skimmed milk and FF full fat.

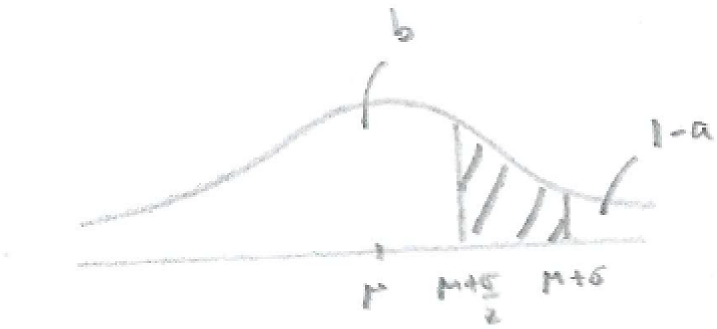
Then $P(Y > 500|Y < 505) = \frac{P(500<Y<505)}{P(Y<505)}$

$P(Y < 505) = P(SM)P(Y < 505|SM) + P(FF)P(Y < 505|FF)$

$= 0.6b + 0.4a$

Similarly, $P(500 < Y < 505)$

$= 0.6(b - 0.5) + 0.4P(\mu + \frac{\sigma}{2} < X < \mu + \sigma)$



$= 0.6(b - 0.5) + 0.4(1 - [b + 1 - a])$ [see diagram above]

$= 0.6(b - 0.5) + 0.4(a - b)$

$= 0.2b + 0.4a - 0.3$

Hence $P(Y > 500|Y < 505) = \frac{0.2b+0.4a-0.3}{0.6b+0.4a} = \frac{2b+4a-3}{6b+4a}$

(b)(ii) $P(Y \leq 505) = 0.7 \Rightarrow 0.6b + 0.4a = 0.7$ (from (b)(i)) (1)

$P(FF|Y \geq 495) = \frac{1}{3}$

$\Rightarrow \frac{P(FF\ \&\ Y\geq 495)}{P(Y\geq 495)} = \frac{1}{3}$ (2)

$P(FF \,\&\, Y \geq 495) = 0.4(0.5) = 0.2$

$P(Y \geq 495) = 0.6P\left(X \geq \mu - \frac{\sigma}{2}\right) + 0.4(0.5)$

$= 0.6b + 0.2$

Then (2) $\Rightarrow \frac{0.2}{0.6b+0.2} = \frac{1}{3}$

$\Rightarrow 0.6 = 0.6b + 0.2$

$\Rightarrow 0.4 = 0.6b \Rightarrow b = \frac{2}{3}$

Then, from (1), $0.4a = 0.7 - 0.6\left(\frac{2}{3}\right) = 0.3 \Rightarrow a = \frac{3}{4}$