STEP 2004, P1, Q14 - Solution (4 pages; 24/7/25)

14 Three pirates are sharing out the contents of a treasure chest containing n gold coins and 2 lead coins. The first pirate takes out coins one at a time until he takes out one of the lead coins. The second pirate then takes out coins one at a time until she draws the second lead coin. The third pirate takes out all the gold coins remaining in the chest.

Find:

(i) the probability that the first pirate will have some gold coins;

(ii) the probability that the second pirate will have some gold coins;

(iii) the probability that all three pirates will have some gold coins.

Solution

(i) Prob. $= P(\text{1st coin is Gold}) = \frac{n}{n+2}$

(ii) The 2nd pirate will **not** receive any Gold coins when the pattern is eg $GGGLLGG$ (for $n = 5$), where the only restriction on the Ls is that they must be together.

So Prob. $= 1 - \frac{\binom{n+1}{1}}{\binom{n+2}{2}}$, where LL is treated as a single item in the numerator;

$= 1 - \frac{n+1}{\left[\frac{(n+2)(n+1)}{2}\right]} = 1 - \frac{2}{n+2} = \frac{n}{n+2}$

[In fact, by symmetry, for every combination where the 1st pirate receives p gold coins and the 2nd pirate receives q gold coins, there will be a corresponding combination where the 1st pirate receives q gold coins and the 2nd pirate receives p gold coins, and so the probability distributions of numbers of gold coins will be the same for pirates 1 and 2, and similarly for pirates 1 and 3; ie all three pirates have the same probability distribution, and so the probability that each receives some gold coins will be the same.

(Also, by symmetry, the number of Gs before the 1st L will have the same probability distribution as the number of Gs after the 2nd L, so that the 1st and 3rd pirates must have the same probability distribution.)]

(iii) We require patterns of the form $GGLGLGG$ (for $n = 5$); ie which start and end with a G, and where the Ls are not together.

The number of these is $\binom{n}{2}$ [the number of ways of choosing the Gs & Ls, apart from the 1st and last Gs] less the ones where the Ls are together: $\binom{n-1}{1}$ (again, treating the LLs as a single item).

So Prob. $= \dfrac{\binom{n}{2}-\binom{n-1}{1}}{\binom{n+2}{2}} = \dfrac{\left[\frac{n(n-1)}{2}\right]-(n-1)}{\left[\frac{(n+2)(n+1)}{2}\right]}$

$= \dfrac{n(n-1)-2(n-1)}{(n+2)(n+1)} = \dfrac{(n-1)(n-2)}{(n+1)(n+2)}$

Check

Consider a Venn Diagram with circles 1, 2 & 3 representing the events that pirates 1, 2 & 3, respectively receive some Gold coins.

Then for (iii) we need $P(1 \cap 2 \cap 3)$.

And $P(1 \cup 2 \cup 3) = P(1) + P(2) + P(3) - P(1 \cap 2) - P(1 \cap 3)$

$-P(2 \cap 3) + P(1 \cap 2 \cap 3)$ (*)

Now, $P(1 \cup 2 \cup 3)$ is clearly 1.

$P(1) = \frac{n}{n+2}$, from (i);

$P(2) = \frac{n}{n+2}$, from (ii)

The 3rd pirate will not receive any Gold coins when the last coin is Lead. So $P(3) = 1 - \frac{2}{n+2} = \frac{n}{n+2}$

For $P(1 \cap 2)$, we require patterns of the form $GGLGLGG$ or

$GGLGGG$L (for $n = 5$); ie starting with a G, with the Ls not together, but where the last coin can be a G or an L.

So $P(1 \cap 2) = \dfrac{\binom{n+1}{2}-\binom{n}{1}}{\binom{n+2}{2}} = \dfrac{\left[\frac{(n+1)n}{2}\right]-n}{\left[\frac{(n+2)(n+1)}{2}\right]}$

$= \dfrac{(n+1)n-2n}{(n+2)(n+1)} = \dfrac{n(n-1)}{(n+1)(n+2)}$

For $P(1 \cap 3)$, we require patterns of the form $GGLGLGG$ or $GGLLGGG$ (for $n = 5$); ie starting and ending with a G, where the Ls may or may not be together

So $P(1 \cap 3) = \frac{\binom{n}{2}}{\binom{n+2}{2}} = \frac{\left[\frac{n(n-1)}{2}\right]}{\left[\frac{(n+2)(n+1)}{2}\right]}$

$= \frac{n(n-1)}{(n+1)(n+2)}$ again.

For $P(2 \cap 3)$, we require patterns of the form $GGLGLGG$ or $LGGGLGG$ (for $n = 5$); ie ending with a G, with the Ls not together, but where the 1st coin can be a G or an L.

So $P(2 \cap 3) = \frac{\binom{n+1}{2} - \binom{n}{1}}{\binom{n+2}{2}}$, as for $P(1 \cap 2)$, which equals $\frac{n(n-1)}{(n+1)(n+2)}$ again.

Therefore, from (*):

$1 = \frac{3n}{n+2} - \frac{3n(n-1)}{(n+1)(n+2)} + P(1 \cap 2 \cap 3)$,

so that $P(1 \cap 2 \cap 3) = \frac{(n+1)(n+2) - 3n(n+1) + 3n(n-1)}{(n+1)(n+2)}$

$= \frac{n^2 - 3n + 2}{(n+1)(n+2)} = \frac{(n-1)(n-2)}{(n+1)(n+2)}$

[The above symmetry argument also applies here, to justify the fact that $P(1 \cap 2) = P(1 \cap 3) = P(2 \cap 3)$.]

[Interestingly, the official Hints & Answers don't make any mention of symmetry, so maybe the examiners wouldn't be convinced by this argument.]