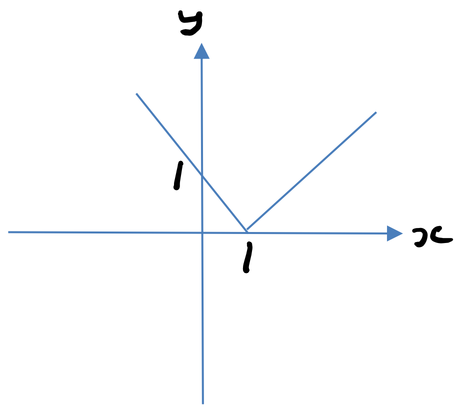
STEP 2003, Paper 2, Q6 - Solution (3 pages; 2/4/24)

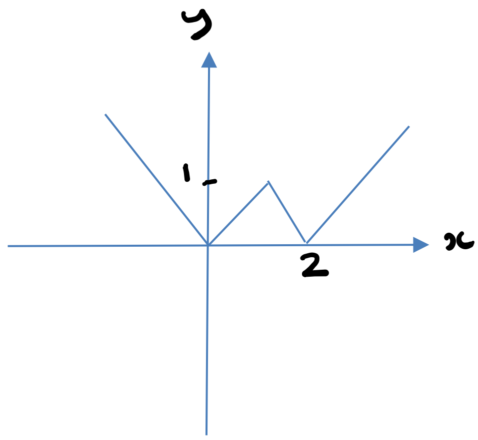
1st Part

$g_1(x) = f(x) = |x - 1|$:



$g_2(x) = |g_1(x) - 1|$

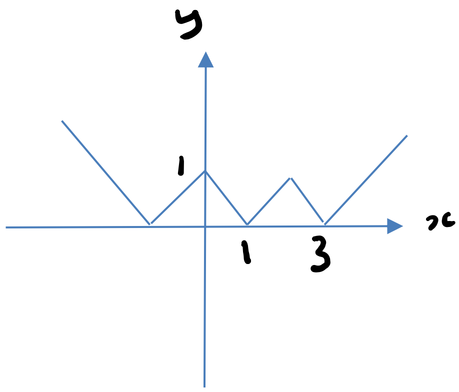
This can be obtained from $g_1(x)$ by translating 1 down, and reflecting in the x-axis whenever $g_1(x) - 1 < 0$



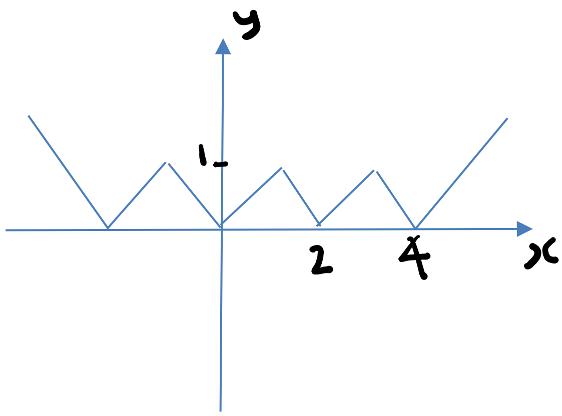
$g_3(x) = |g_2(x) - 1|$

Again, this can be obtained from $g_2(x)$ by translating 1 down, and reflecting in the x-axis whenever $g_2(x) - 1 < 0$ (or reflecting in $y = \frac{1}{2}$, as then the new function is $1 - g_2(x)$).

[Because of the symmetry of $g_2(x)$, $g_3(x)$ will also have symmetry about $x = 1$]

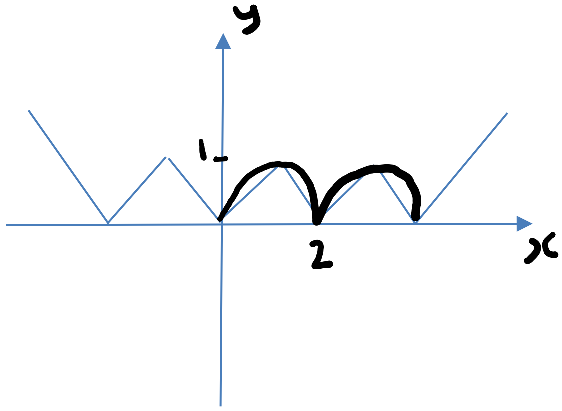


Similarly for $g_4(x)$:



2nd Part

The graph of $\left|\sin\left(\frac{\pi}{2}x\right)\right|$ is shown in bold below, with $g_4(x)$:



From the graphs in the 1st Part, the number of hoops in the interval $[0, n]$ is seen to be $\frac{n}{2}$.

[The 'show' in the question, and the general spirit of STEP questions suggest that this should be enough. Otherwise we could try to develop a proof by induction.]

Therefore $\int_0^n |sin\left(\frac{\pi}{2}x\right)| - g_n(x)\, dx$

$= \frac{n}{2}.\, 2\int_0^1 \sin\left(\frac{\pi}{2}x\right) - x\, dx$

$= n[-\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right) - \frac{1}{2}x^2]_0^1$

$= n[-\frac{1}{2} - \left(-\frac{2}{\pi}\right)]$

$= \frac{2n}{\pi} - \frac{n}{2}$, as required