STEP 2001, Paper 2, Q11 (4 pages; 30/3/25)

11 A two-stage missile is projected from a point A on the ground with horizontal and vertical velocity components u and v, respectively. When it reaches the highest point of its trajectory an internal explosion causes it to break up into two fragments. Immediately after this explosion one of these fragments, P, begins to move vertically upwards with speed v_e, but retains the previous horizontal velocity. Show that P will hit the ground at a distance R from A given by

$$\frac{gR}{u} = v + v_e + \sqrt{(v_e^2 + v^2)}.$$

It is required that the range R should be greater than a certain distance D (where $D > 2uv/g$). Show that this requirement is satisfied if

$$v_e > \frac{gD}{2u}\left(\frac{gD - 2uv}{gD - uv}\right).$$

[The effect of air resistance is to be neglected.]

1st Part

Let d_1& d_2 be the horizontal distances covered before and after the explosion (so that $R = d_1 + d_2$).

Also, let T_1& T_2 be the times taken to cover these distances.

Then, if H is the maximum height reached by the missile (before the explosion), the suvat equation $'v^2 = u^2 + 2as'$ applied to the vertical component of motion gives

$0 = v^2 + 2(-g)H$, so that $H = \frac{v^2}{2g}$ (1)

Also, $'v = u + at'$ (again, vertically) gives $0 = v + (-g)T_1$,

and (horizontally) $d_1 = uT_1 = u\frac{v}{g}$ (2)

Then, after the explosion, $'s = s_0 + ut + \frac{1}{2}at^2{}'$ (vertically) gives (with upwards as the positive direction):

$0 = H + v_e T_2 + \frac{1}{2}(-g){T_2}^2$ (*)

and (horizonally) $d_2 = uT_2$ (3)

Then, from (*), $g{T_2}^2 - 2v_e T_2 - 2(\frac{v^2}{2g}) = 0$, from (1),

so that $T_2 = \frac{2v_e + \sqrt{4{v_e}^2 + 4v^2}}{2g}$ (taking the positive root)

$= \frac{v_e + \sqrt{{v_e}^2 + v^2}}{g}$ (4)

And so $\frac{gR}{u} = \frac{g}{u}(d_1 + d_2) = v + gT_2$, from (2) & (3),

$= v + v_e + \sqrt{{v_e}^2 + v^2}$, from (4), as required.

2nd Part

[The fact that $D > \frac{2uv}{g}$ means that both $Dg - 2uv$ & $Dg - uv$ are positive, so that the lower limit for v_e that we are being asked to establish is a positive value. (If it were negative, it would automatically be satisfied.)]

The requirement that $R > D$ is equivalent to $\frac{gR}{u} > \frac{gD}{u}$,

or $\frac{gR}{u} > D'$, where $D' = \frac{gD}{u}$ (1)

And $v_e > \frac{gD}{2u}(\frac{gD-2uv}{gD-uv})$ is equivalent to $v_e > \frac{D'}{2}(\frac{D'-2v}{D'-v})$ (2)

Also, from the 1st Part, (1) is equivalent to

$v + v_e + \sqrt{{v_e}^2 + v^2} > D'$ $(1')$

Result to prove: (2) $\Rightarrow$ $(1')$

Now, $(1') \Leftrightarrow \sqrt{{v_e}^2 + v^2} > D' - v - v_e$ (A)

Consider (B): ${v_e}^2 + v^2 > (D' - v - v_e)^2$

As $\sqrt{{v_e}^2 + v^2} \geq 0$, (B) $\Rightarrow (A)$ [whether or not $D' - v - v_e > 0$]

[though (B) $\not\Leftarrow (A)$, as it may be the case that $D' - v - v_e > 0$]

So, result to prove is now: (2) $\Rightarrow (B)$

(then, from the above, (2) $\Rightarrow (B) \Rightarrow (A) \Rightarrow (1')$).

where (2) is: $v_e > \frac{D'}{2}(\frac{D'-2v}{D'-v})$,

and (B) is ${v_e}^2 + v^2 > (D' - v - v_e)^2$, which is equivalent to

$0 > (D')^2 - 2D'v - 2D'v_e + 2vv_e$ (B')

And (2): $v_e > \frac{D'}{2}(\frac{D'-2v}{D'-v})$ is equivalent to

$2v_e(D'-v) > (D')^2 - 2D'v$

(as $D'-v > 0$, given that $D > \frac{2uv}{g}$, so that $D' = \frac{gD}{u} > 2v > v$),

which is equivalent to (B').

Thus, in particular, $(2) \Rightarrow (B') \equiv (B)$, as required.