STEP 2001, P1, Q2 - Solution (3 pages; 13/8/25)

2 Solve the inequalities

(i) $1 + 2x - x^2 > 2/x \qquad (x \neq 0)$,

(ii) $\sqrt{(3x+10)} > 2 + \sqrt{(x+4)} \qquad (x \geqslant -10/3)$.

Solution

(i) If $x > 0$, then $1 + 2x - x^2 > \frac{2}{x} \Leftrightarrow x + 2x^2 - x^3 > 2$,

ie $x^3 - 2x^2 - x + 2 < 0$,

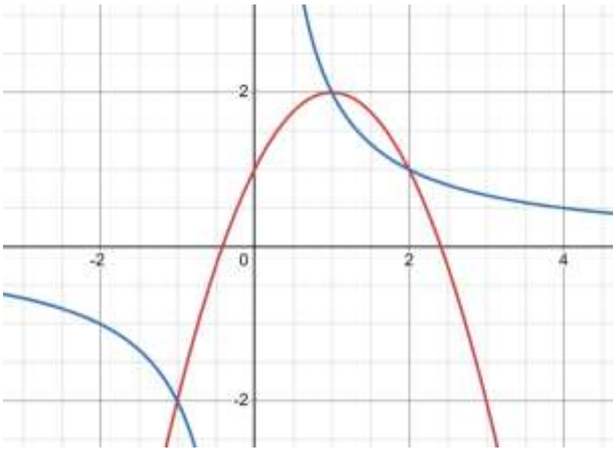
ie $(x-1)(x^2 - x - 2) < 0$,

ie $(x-1)(x-2)(x+1) < 0$ ($\&\ x > 0$),

so that (from a sketch of the cubic on the LHS) $1 < x < 2$

If instead $x < 0$, $(x-1)(x-2)(x+1) > 0$, so that $-1 < x < 0$

Thus, the solution is that either $-1 < x < 0$ or $1 < x < 2$.



(ii) $\sqrt{3x+10} > 2 + \sqrt{x+4}$

$\Leftrightarrow 3x + 10 > 4 + (x+4) + 4\sqrt{x+4}$,

(since $a^2 > b^2 \Rightarrow a > b$ if $a > 0$, and it is given that

$\sqrt{3x+10} > 2 + \sqrt{x+4} \geq 0$, so that $\sqrt{3x+10} > 0$)

ie $2x + 2 > 4\sqrt{x+4}$; $x + 1 > 2\sqrt{x+4}$, which implies the restriction $x + 1 > 0$; ie $x > -1$

$\Leftrightarrow (x + 1)^2 > 4(x + 4)$,

(Again, $a^2 > b^2 \Rightarrow a > b$ if $a > 0$, and it is given that

$x + 1 > 2\sqrt{x + 4} \geq 0$, so that $x + 1 > 0$)

ie $x^2 - 2x - 15 > 0$,

ie $(x - 5)(x + 3) > 0$,

so that $-\frac{10}{3} \leq x < -3$ *or* $x > 5$, and also $x > -1$

Thus the solution is $x > 5$.

[As a check, it is possible to sketch the curves $y = \sqrt{3x + 10}$ and

$y = 2 + \sqrt{x + 4}$, by carrying out the necessary transformations;

see below.]