STEP 1989, Paper 3, Q9 (4 pages; 18/4/25)

9 Obtain the sum to infinity of each of the following series.

(i) $1+\frac{2}{2}+\frac{3}{2^2}+\frac{4}{2^3}+\cdots+\frac{r}{2^{r-1}}+\cdots\,;$

(ii) $1+\frac{1}{2}\times\frac{1}{2}+\frac{1}{3}\times\frac{1}{2^2}+\cdots+\frac{1}{r}\times\frac{1}{2^{r-1}}+\cdots\,;$

(iii) $\frac{1\times 3}{2!}\times\frac{1}{3}+\frac{1\times 3\times 5}{3!}\frac{1}{3^2}+\cdots+\frac{1\times 3\times\cdots\times(2k-1)}{k!}\times\frac{1}{3^{k-1}}+\cdots\,.$

[Questions of convergence need not be considered.]

Solution

[Not verified against any other solution.]

(i) Consider $\frac{d}{dx}\sum_{r=1}^{\infty} x^r = \sum_{r=1}^{\infty} \frac{d}{dx}(x^r) = \sum_{r=1}^{\infty} rx^{r-1}$, with $x = \frac{1}{2}$,

which is the required sum.

The sum is therefore $\frac{d}{dx}\sum_{r=1}^{\infty} x^r \ \Big|_{x=\frac{1}{2}}$

and $\frac{d}{dx}\sum_{r=1}^{\infty} x^r = \frac{d}{dx}\ (\frac{x}{1-x})$ (provided $|x| < 1$)

$= \frac{(1-x)(1)-x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$,

so that the sum is $\frac{1}{(1-(\frac{1}{2}))^2} = 4$

(ii) Consider $\int \sum_{r=1}^{\infty} x^r \ dx = \sum_{r=1}^{\infty} \int x^r\, dx = \sum_{r=1}^{\infty} \frac{1}{r+1} x^{r+1}$

Writing $R = r + 1$ gives $\sum_{R=2}^{\infty} \frac{1}{R} x^R = x\sum_{R=2}^{\infty} \frac{1}{R} x^{R-1}$,

so that $\frac{1}{x}\int \sum_{r=1}^{\infty} x^r = \sum_{r=2}^{\infty} \frac{1}{r} x^{r-1} = \left(\sum_{r=1}^{\infty} \frac{1}{r} x^{r-1}\right) - 1$

and hence $\sum_{r=1}^{\infty} \frac{1}{r} x^{r-1} = 1 + \frac{1}{x}\int \frac{x}{1-x}\ dx$ (again, provided $|x| < 1$)

$= 1 + \frac{1}{x}\int \frac{x-1}{1-x} + \frac{1}{1-x}\ dx = 1 + \frac{1}{x}(-x - \ln|1-x| + C)$

or $x\sum_{r=1}^{\infty} \frac{1}{r} x^{r-1} = x - x - \ln|1-x| + C$

Then, setting $x = 0$ gives $C = 0$,

so that $\sum_{r=1}^{\infty} \frac{1}{r} x^{r-1} = -\frac{\ln\ |1-x|}{x}$

Setting $x = \frac{1}{2}$, the required sum is $= -\frac{\ln\left|1-\frac{1}{2}\right|}{\left(\frac{1}{2}\right)} = 2ln2$

(iii) [A promising-looking approach, which unfortunately doesn't seem to lead anywhere, is to note that

$\frac{1\times3\times\ldots\times(2k-1)}{k!}\times\frac{1}{3^{k-1}}=\frac{(2k)!}{k!(2\times4\times6\times\ldots\times2k)}\times\frac{1}{3^{k-1}}$

$\frac{(2k)!}{k!k!2^k}\times\frac{1}{3^{k-1}}=3\binom{2k}{k}(\frac{1}{6})^k$. Instead though, the $\frac{1}{k!}$ suggests a Maclaurin expansion, and it could be observed that the successive derivatives of $x^{-\frac{1}{2}}$ produce the factors $\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\ldots$ The power of 2 generated by this can be incorporated into the power of x in the expansion. However, there are two snags: the alternating signs, and the fact that the coefficients of the powers of x will be $f(0), f'(0), \ldots$, which won't be defined if $f(x)=x^{-\frac{1}{2}}$.

But both of these problems can be removed by having $f(x)=(1-x)^{-\frac{1}{2}}$, and we can of course just use the formula for a general Binomial expansion, rather than having to obtain a Maclaurin expansion from first principles.]

Consider $f(x)=(1-x)^{-\frac{1}{2}}=1+\left(-\frac{1}{2}\right)(-x)+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(-x)^2$

$+\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}(-x)^3+\cdots$ (provided $|x|<1$)

$=1+\frac{x}{2}+\frac{1\times3}{2!}(\frac{x}{2})^2+\frac{1\times3\times5}{3!}(\frac{x}{2})^3+\cdots+\frac{1\times3\times\ldots\times(2k-1)}{k!}\left(\frac{x}{2}\right)^k+\cdots$

Then $[(1-x)^{-\frac{1}{2}}-\left(1+\frac{x}{2}\right)]\div(\frac{x}{2})$

$=\frac{1\times3}{2!}\left(\frac{x}{2}\right)+\frac{1\times3\times5}{3!}\left(\frac{x}{2}\right)^2+\cdots+\frac{1\times3\times\ldots\times(2k-1)}{k!}\left(\frac{x}{2}\right)^{k-1}+\cdots$

Then setting $\frac{x}{2}=\frac{1}{3}$; ie $x=\frac{2}{3}$, the RHS is the required sum, and so the sum is $[(1-\frac{2}{3})^{-\frac{1}{2}}-\left(1+\frac{1}{3}\right)]\div(\frac{1}{3})$

$=3\left(\sqrt{3}-\frac{4}{3}\right)=3\sqrt{3}-4$