Motion of a particle in a rotating frame of reference

(27 pages; 29/1/24)

Contents

(A) Frames of reference

(A/1) This note concerns the motion of a particle moving with respect to a frame of reference (or 'frame', for short) $Oxyz$ that is itself rotating relative to a fixed frame $OXYZ$. Both frames of reference have mutually perpendicular axes, and share the same Origin. Let $\underline{i}$, $\underline{j}$ and $\underline{k}$ be unit vectors in the rotating frame, in the x, y & z directions, and $\underline{I}$, $\underline{J}$ and $\underline{K}$ unit vectors in the fixed frame, in the X, Y & Z directions.

(A/2) The fixed frame is assumed to be inertial – where an inertial frame can be defined to be a frame in which Newton's 1st Law holds. Newton's 2nd Law then holds in any inertial frame. The main aim is to derive an equation of motion for the particle. Newton's 2nd law is only valid in an inertial (and therefore fixed) frame, and so the equation of motion is initially established for the fixed frame. By invoking 'apparent' (or 'fictitious') forces, it is then possible to come up with an equation that applies in the rotating frame. The most familiar example is of the rotating Earth, where centrifugal and Coriolis forces appear in the equation of motion, from the point of view of an observer on the Earth.

(A/3) There are two situations that can be considered:

Planar case: This is the considerably simplified situation where the particle is constrained to move in a plane. The z axis of the rotating frame is the same as the Z axis of the fixed frame, and the rotating frame rotates about the z axis. The angular velocity of the rotating frame will therefore be of the form $\underline{\omega}(t) = \omega(t)\underline{k}$.

As indicated by the notation, $\omega(t)$ may vary with time, but it may be constant.

General case: Now the particle can move in 3 dimensions, and the axis of rotation of the rotating frame of reference can be in any direction, and can vary with time. The only constraint is that the fixed and rotating frames share an Origin.

The angular velocity of the rotating frame will now be of the form $\underline{\omega}(t) = \omega_1(t)\underline{i} + \omega_2(t)\underline{j} + \omega_3(t)\underline{k}$. It may seem odd that $\underline{\omega}(t)$ is expressed in terms of unit vectors in the rotating frame, when the idea of the angular velocity is to represent the rotation relative to the fixed frame. This point is discussed in (A/4).

It can help to consider a rigid body fixed in the rotating frame. In the case of a cuboid, the $\underline{x}$, $\underline{y}$ and $\underline{z}$ axes could be in the directions of 3 mutually perpendicular edges meeting at a corner of the cuboid. The cuboid would, at any moment, have an overall rotation made up of rotations about these 3 edges.

(A/4) From the point of view of an observer in the fixed frame, $\underline{i}(t)$, $\underline{j}(t)$ and $\underline{k}(t)$ vary with time (as indicated by the notation), but they can (in principle) be expressed in terms of $\underline{I}$, $\underline{J}$ and $\underline{K}$.

[See (C1/2) and (C2/1). In practice though, it may not be feasible to do this – especially for non-planar cases.]

This means that any vector (including the position, velocity and acceleration of a particle, as well as the angular velocity of the rotating frame (or a rigid body fixed in the frame)) can (in principle) be expressed in terms of either the fixed axes, or the rotating axes. Thus the fact that quantities are expressed in terms of the rotating axes does not stop them from being quantities viewed in the fixed frame. (As mentioned, equations need to be set up in the inertial fixed frame, where Newton's 2nd Law applies.)

In particular, the angular velocity $\underline{\omega}(t)$ represents how the rotating frame (or a rigid body in that frame) is rotating relative to the fixed frame, and yet $\underline{\omega}(t)$ will invariably be given in terms of $\underline{i}$, $\underline{j}$ and $\underline{k}$.

(B) Position & velocity vectors

[Note: In the Planar case, the 3rd dimension won't be needed.]

(B/1) Consider a particle, represented by a point P, moving relative to the rotating frame (as well as to the fixed frame). Its position vector can be represented in several ways:

(i) With respect to an observer in the rotating frame,

$\underline{r}_R(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$ (where $\underline{i}$, $\underline{j}$ and $\underline{k}$ appear to be fixed)

(ii) With respect to an observer in the fixed frame,

$\underline{r}_F(t) = x(t)\underline{i}(t) + y(t)\underline{j}(t) + z(t)\underline{k}(t)$,

and in principle $\underline{i}(t), \underline{j}(t)$ and $\underline{k}(t)$ can be written in terms of $\underline{I}$, $\underline{J}$ and $\underline{K}$, to give:

(iii) $\underline{r}_F(t) = X(t)\underline{I} + Y(t)\underline{J} + Z(t)\underline{K}$

Note: Textbooks tend to just use the symbol $\underline{r}$ for the position vector for all of the above forms. In each case, we are establishing the particle's position in relation to the Origin, and the three forms are algebraically equivalent. (Were the rotating frame to have a different Origin from the fixed frame, then $\underline{r}_R(t)$ wouldn't be equal to $\underline{r}_F(t)$.)

(B/2) A velocity vector for the particle can be established, in the following forms:

(i) With respect to to an observer in the rotating frame,

$\underline{r}_R(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$,

and so $\dot{\underline{r}}_R(t) = \dot{x}(t)\underline{i} + \dot{y}(t)\underline{j} + \dot{z}(t)\underline{k}$,

as $\underline{i}$, $\underline{j}$ and $\underline{k}$ are fixed from the point of view of this observer.

(ii) With respect to the fixed frame,

$\underline{r}_F(t) = x(t)\underline{i}(t) + y(t)\underline{j}(t) + z(t)\underline{k}(t)$

and, by the product rule for differentiation,

$$\dot{\underline{r}}_F(t) = \dot{x}(t)\underline{i}(t) + \dot{y}(t)\underline{j}(t) + \dot{z}(t)\underline{k}(t)$$

$$+ x(t)\frac{d}{dt}\underline{i}(t) + y(t)\frac{d}{dt}\underline{j}(t) + z(t)\frac{d}{dt}\underline{k}(\mathrm{t}) \qquad (1)$$

(iii) Alternatively, where $\underline{i}(t)$ and $\underline{j}(t)$ can conveniently be expressed in terms of $\underline{I}$ and $\underline{J}$, we can obtain

$\underline{r}_F(t) = X(t)\underline{I} + Y(t)\underline{J} + Z(t)\underline{K}$,

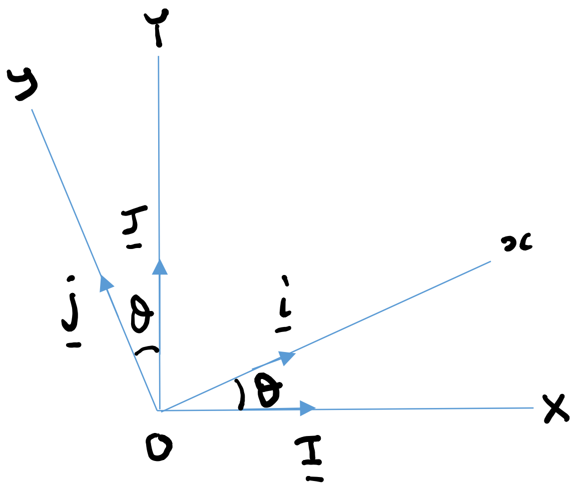
and $\dot{\underline{r}}_F(t) = \dot{X}(t)\,\underline{I} + \dot{Y}(t)\underline{J} + \dot{Z}(t)\underline{K}$ (as $\underline{I}$, $\underline{J}$ and $\underline{K}$ are fixed).

(C) Angular velocity of the rotating frame

(C1) Planar case

(C1/1) In this simplified situation, we assume that:

(i) The particle's motion takes place in the OXY and Oxy planes, and (ii) The direction z is fixed, and is the same as that of Z.



Referring to the diagram,

$$\underline{i}(t) = cos\theta(t)\underline{I} + sin\theta(t)\underline{J} \text{ and } \underline{j}(t) = -sin\theta(t)\underline{I} + cos\theta(t)\underline{J} \quad (2)$$

(C1/2) At a given time t, it is possible to establish θ (as in the diagram above) from the history of the rotation of the frame (or the body). Once θ has been established, the unit vectors in the rotating frame, $\underline{i}(t)$ & $\underline{j}(t)$, can then (if necessary) be expressed in terms of the unit vectors in the fixed frame, $\underline{I}$ & $\underline{J}$.

(C1/3) Let $\omega(t) = \dot{\theta}(t)$ be the rate at which Oxy rotates relative to OXY, about the axis $Z = z$ (where, in general, $\omega(t)$ can vary with time). $\omega(t)$ is referred to as the angular speed of the frame (or body), and the angular velocity in the planar case is

$\underline{\omega}(t) = \omega(t)\underline{k}$.

Note: If the body rotates at a constant rate ω, then $\theta = \omega t$.

(C1/4) From (2), $\frac{d}{dt}\underline{i}(t) = -sin\theta(t).\dot{\theta}(t)\,\underline{I} + cos\theta(t).\dot{\theta}(t)\,\underline{J}$

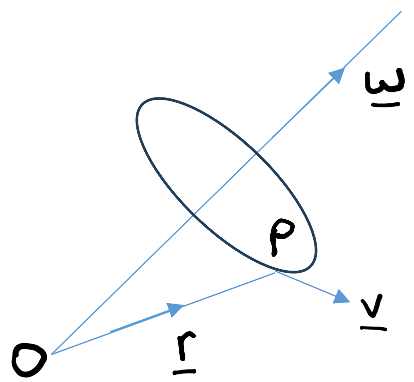
$= \dot{\theta}(t)\,\underline{j}(t)$ or $\omega(t)\underline{j}(t)$

and $\frac{d}{dt}\underline{j}(t) = -cos\theta(t).\dot{\theta}(t)\underline{I} - sin\theta(t).\dot{\theta}(t)\,\underline{J}$

$= -\dot{\theta}(t)\,\underline{i}(t)$ or $-\omega(t)\underline{i}(t)$

$$\frac{d}{dt}\underline{i}(t) = \omega(t)\underline{j}(t) \;\; ; \;\; \frac{d}{dt}\underline{j}(t) = -\omega(t)\underline{i}(t) \qquad (3)$$

(C2) General case



(C2/1) Consider a rigid body that is fixed in the rotating frame – for example, a cuboid, as discussed earlier. We will show that a vector $\underline{\omega}_F(t)$ can be found such that every point of the body is instantaneously rotating about an axis through the Origin in the direction of $\underline{\omega}_F(t)$, with this axis being momentarily at rest, relative to the fixed frame $OXYZ$.

[Note: The notation $\underline{\omega}_F(t)$ has been used, rather than $\underline{\omega}(t)$ (as in the Planar case), to indicate that the angular velocity is relative to the fixed frame. In the Planar case, $\underline{\omega}(t) = \omega(t)\underline{k}$ is relative to the rotating frame (as indicated by the constant $\underline{k}$), but the direction of $\underline{k}$ is always $\underline{K}$. Later on, we will write $\underline{\omega}_F(t) = \omega(t)\underline{k}(t)$, with $\underline{k}(t) = \underline{K}$, to indicate that $\underline{\omega}_F(t)$ is also relative to the fixed frame.]

In general, the direction of the axis can change with time. The orientation of the rotating $\underline{i}$, $\underline{j}$, $\underline{k}$ frame relative to the fixed $\underline{I}$, $\underline{J}$, $\underline{K}$ frame at any specific moment will be determined by the history of the angular velocity vector $\underline{\omega}_F(t)$; ie knowledge of $\underline{\omega}_F(t)$ will (in principle) completely specify $\underline{i}(t)$, $\underline{j}(t)$, $\underline{k}(t)$ in terms of $\underline{I}$, $\underline{J}$, $\underline{K}$, at any specific moment.

(C2/2) First of all, we can establish that $\frac{d\underline{i}(t)}{dt} = \underline{\omega}_F(t) \times \underline{i}(\mathrm{t})$, for a suitable $\underline{\omega}_F(t)$, and similarly for $\frac{d\underline{j}(t)}{dt}$ & $\frac{d\underline{k}(t)}{dt}$

Proof

First of all, $\underline{i}(t).\underline{i}(t) = 1$, and differentiating wrt time gives

$\frac{d\underline{i}(t)}{dt}.\underline{i}(t) + \underline{i}(t).\frac{d\underline{i}(t)}{dt} = 0$, so that $\underline{i}(t).\frac{d\underline{i}(t)}{dt} = 0$

This means that $\frac{d\underline{i}(t)}{dt}$ is perpendicular to $\underline{i}(t)$, and can therefore be written as $\beta_1(t)\underline{j}(t) + \gamma_1(t)\underline{k}(t)$ (*)

Similarly, $\frac{d\underline{j}(t)}{dt} = \gamma_2(t)\underline{k}(t) + \alpha_2(t)\underline{i}(t)$

and $\frac{d\underline{k}(t)}{dt} = \alpha_3(t)\underline{i}(t) + \beta_3(t)\underline{j}(t)$

Also, $\underline{i}(t).\underline{j}(t) = 0$, and differentiating then gives

$\frac{d\underline{i}(t)}{dt}.\underline{j}(t) + \underline{i}(t).\frac{d\underline{j}(t)}{dt} = 0$

Hence,

$[\beta_1(t)\underline{j}(t) + \gamma_1(t)\underline{k}(t)].\underline{j}(t) + \underline{i}(t).\left[\gamma_2(t)\underline{k}(t) + \alpha_2(t)\underline{i}(t)\right] = 0$,

so that $\beta_1(t) + \alpha_2(t) = 0$, and we can write

$\omega_3(t) = \beta_1(t) = -\alpha_2(t)$ (ie defining $\omega_3(t)$ in this way)

Similarly, $\omega_1(t) = \gamma_2(t) = -\beta_3(t)$ and $\omega_2(t) = \alpha_3(t) = -\gamma_1(t)$,

and so $\frac{d\underline{i}(t)}{dt} = \beta_1(t)\underline{j}(t) + \gamma_1(t)\underline{k}(t)$ (from (*))

$= \omega_3(t)\underline{j}(t) - \omega_2(t)\underline{k}(t)$,

and similarly, $\frac{d\underline{j}(t)}{dt} = \omega_1(t)\underline{k}(t) - \omega_3(t)\underline{i}(t)$

and $\frac{d\underline{k}(t)}{dt} = \omega_2(t)\underline{i}(t) - \omega_1(t)\underline{j}(t)$.

Then if we define $\underline{\omega}_F(t) = \omega_1(t)\underline{i}(t) + \omega_2(t)\underline{j}(t) + \omega_3(t)\underline{k}(t)$,

it follows that $\underline{\omega}_F(t) \times \underline{i}(t) = -\omega_2(t)\underline{k}(t) + \omega_3(t)\underline{j}(t) = \frac{d\underline{i}(t)}{dt}$,

and similarly $\underline{\omega}_F(t) \times \underline{j}(t) = \frac{d\underline{j}(t)}{dt}$

and $\underline{\omega}_F(t) \times \underline{k}(t) = \frac{d\underline{k}(t)}{dt}$, as required.

Thus $\frac{d\underline{i}(t)}{dt} = \underline{\omega}_F(t) \times \underline{i}(t)$,

$\frac{d\underline{j}(t)}{dt} = \underline{\omega}_F(t) \times \underline{j}(t)$

and $\frac{d\underline{k}(t)}{dt} = \underline{\omega}_F(t) \times \underline{k}(t)$ (4)

(C2/3) We will now show that any point fixed in the rigid body is instantaneously performing circular motion about the the line through the Origin in the direction of $\underline{\omega}_F(t)$.

Now, if $\underline{r}_F(t) = a\underline{i}(t) + b\underline{j}(t) + c\underline{k}(t)$ is a point that is fixed in the body (so that a, b & c are constant), from the point of view of an observer in the fixed frame, then

$\frac{d\underline{r}_F(t)}{dt} = a\frac{d\underline{i}(t)}{dt} + b\frac{d\underline{j}(t)}{dt} + c\frac{d\underline{k}(t)}{dt}$

$= a\underline{\omega}_F(t) \times \underline{i}(t) + b\underline{\omega}_F(t) \times \underline{j}(t) + c\underline{\omega}_F(t) \times \underline{k}(\mathrm{t})$

$= \underline{\omega}_F(t) \times \underline{r}_F(t) \quad (*)$

This shows that the direction of motion of the point (the direction of $\frac{d\underline{r}_F(t)}{dt}$) is perpendicular to both $\underline{r}_F(t)$ and $\underline{\omega}_F(t)$; ie perpendicular to the plane containing $\underline{r}_F(t)$and $\underline{\omega}_F(t)$ (by the definition of the vector product), and the point is therefore performing circular motion about the line through O with direction $\underline{\omega}_F(t)$. Also, (*) implies that $\underline{\omega}_F(t)$ is the angular velocity of the body.

(D) $\frac{d\underline{r}}{dt} = \frac{\delta\underline{r}}{\delta t} + \underline{\omega} \times \underline{r}$

(D1) Planar case

(D1/1) From (3), $\frac{d}{dt}\underline{i}(t) = \omega(t)\,\underline{j}(t)$ and $\frac{d}{dt}\underline{j}(t) = -\omega(t)\,\underline{i}(\mathrm{t})$,

and so, using the form (ii) referred to previously (but in 2D only):

$\dot{\underline{r}}_F(t) = \dot{x}(t)\underline{i}(t) + \dot{y}(t)\underline{j}(t) + x(t)\frac{d}{dt}\underline{i}(t) + y(t)\frac{d}{dt}\underline{j}(t)$

$= \dot{x}(t)\underline{i}(t) + \dot{y}(t)\underline{j}(t) + x(t)\omega(t)\,\underline{j}(t) - y(t)\omega(t)\,\underline{i}(t) \quad (5)$

(D1/2) Now, $\underline{\omega}_F(t) = \omega(t)\underline{k}(t)$ is the angular velocity of the rotating frame, and $\underline{\omega}_F(t) \times \underline{r}_F(t) = \begin{vmatrix} \underline{i}(t) & 0 & x(t) \\ \underline{j}(t) & 0 & y(t) \\ \underline{k}(t) & \omega(t) & 0 \end{vmatrix}$

$= -\omega(t)y(t)\underline{i}(t) - (-\omega(t)x(t))\underline{j}(t)$

Hence, from (5):

$\dot{\underline{r}}_F(t) == \dot{x}(t)\underline{i}(t) + \dot{y}(t)\underline{j}(t) + x(t)\omega(t)\,\underline{j}(t) - y(t)\omega(t)\,\underline{i}(t)$

$= \dot{x}(t)\underline{i}(t) + \dot{y}(t)\underline{j}(t) + \underline{\omega}_F(t) \times \underline{r}_F(t) \quad (6)$

Hence:

$\dot{\underline{r}}_F(t) = \frac{\delta \underline{r}_F}{\delta t} + \underline{\omega}_F(t) \times \underline{r}_F(t) \quad (7),$

where $\underline{r}_F(t) = x(t)\underline{i}(t) + y(t)\underline{j}(t),$

$\frac{\delta \underline{r}_F}{\delta t} = \dot{x}(t)\underline{i}(t) + \dot{y}(t)\underline{j}(t),$ and $\underline{\omega}_F(t) = \omega(t)\underline{k}(t)$

(7) is usually written in textbooks, more concisely, as

$\frac{d\underline{r}}{dt} = \frac{\delta \underline{r}}{\delta t} + \underline{\omega} \times \underline{r} \quad (8)$

Notes

(i) There are two components to the particle's velocity relative to the fixed frame:

(a) its velocity relative to the rotating frame ($\frac{\delta \underline{r}_F}{\delta t}$),

and (b) the tangential velocity of the rotating frame, at the point occupied by the particle, as it rotates about its axis

($\underline{\omega}_F(t) \times \underline{r}_F(t)$)

[Strictly speaking, the velocity relative to the rotating frame is $\dot{x}(t)\underline{i} + \dot{y}(t)\underline{j}$ (rather than $\dot{x}(t)\underline{i}(t) + \dot{y}(t)\underline{j}(t)$), with $\underline{i}$ and $\underline{j}$ being fixed, from the point of view of an observer in the rotating frame.]

(ii) As $\frac{\delta \underline{r}_F}{\delta t}$ will invariably be expressed in terms of the rotating frame (ie $\underline{i}(t)$ and $\underline{j}(t)$), this means that the other terms in the equation will need to be expressed in terms of the rotating frame as well.

Thus, despite being relative to the fixed frame, the two components of the velocity are given in terms of the rotating unit vectors (but, as mentioned, these can in principle be expressed in terms of the fixed unit vectors).

[A possible source of confusion in this topic is the fact that the rotating coordinate system can be thought of as having two roles:

(a) Providing an alternative set of unit vectors that can be used (in place of the fixed unit vectors) when describing motion relative to the fixed frame,

and (b) Indicating how the position, velocity or acceleration of a particle appears to an observer in the rotating frame.]

(iii) In equation (8), $\underline{\omega}$ may vary with time.

(D2) General case

(D2/1) From (4), $\frac{d\underline{i}(t)}{dt} = \underline{\omega}_F(t) \times \underline{i}(t)$ etc, so that

$x(t)\frac{d\underline{i}(t)}{dt} + y(t)\frac{d\underline{j}(t)}{dt} + z(t)\frac{d\underline{k}(t)}{dt}$

$= \underline{\omega}_F(t) \times x(\text{t})\underline{i}(t) + \underline{\omega}_F(t) \times y(\text{t})\underline{j}(t) + \underline{\omega}_F(t) \times z(\text{t})\underline{k}(t)$

$= \underline{\omega}_F(t) \times \underline{r}_F(t)$, and so from (1):

$\dot{\underline{r}}_F(t) = \dot{x}(t)\underline{i}(t) + \dot{y}(t)\underline{j}(t) + \dot{z}(t)\underline{k}(t)$

$+ x(t)\frac{d}{dt}\underline{i}(t) + y(t)\frac{d}{dt}\underline{j}(t) + z(t)\frac{d}{dt}\underline{k}(\text{t})$

$= \dot{x}(t)\underline{i}(t) + \dot{y}(t)\underline{j}(t) + \dot{z}(t)\underline{k}(t) + \underline{\omega}_F(t) \times \underline{r}_F(t)$

ie $\dot{\underline{r}}_F(t) = \frac{\delta \underline{r}_F}{\delta t} + \underline{\omega}_F(t) \times \underline{r}_F(t), \quad (7')$

where $\frac{\delta \underline{r}_F}{\delta t} = \dot{x}(t)\underline{i}(t) + \dot{y}(t)\underline{j}(t) + \dot{z}(t)\underline{k}(t)$

As before, this is usually written as $\frac{d\underline{r}}{dt} = \frac{\delta \underline{r}}{\delta t} + \underline{\omega} \times \underline{r}$. $\quad (8')$

Notes

(i) When the direction of $\underline{\omega}_F(t)$ varies, it is not usually feasible to convert $\underline{i}(t)$, $\underline{j}(t)$ and $\underline{k}(t)$ into expressions involving $\underline{I}$, $\underline{J}$ and $\underline{K}$. But, as already mentioned, the orientation of the rotating frame relative to the fixed frame at any specific moment will be determined by the history of the angular velocity vector $\underline{\omega}_F(t)$, and so knowledge of $\underline{\omega}_F(t)$ will (in principle) completely specify $\underline{i}(t)$, $\underline{j}(t)$, $\underline{k}(t)$ in terms of $\underline{I}$, $\underline{J}$, $\underline{K}$, at any specific moment. Fortunately, it is usually more convenient anyway to establish results in terms of the moving unit vectors.

(D3) Application to a general vector

The result (8) or (8′) can in fact be established for any vector of the form $\underline{u}_F(t) = u_x(t)\underline{i}(t) + u_y(t)\underline{j}(t) + u_z(t)\underline{k}(t)$, so that

$$\dot{\underline{u}}_F(t) = \dot{u}_x(t)\underline{i}(t) + \dot{u}_y(t)\underline{j}(t) + \dot{u}_z(t)\underline{k}(t) + \underline{\omega}_F(t) \times \underline{u}_F(t)$$

or $\dot{\underline{u}}_F(t) = \frac{\delta \underline{u}_F(t)}{\delta t} + \underline{\omega}_F(t) \times \underline{u}_F(t)$ (9)

[usually written as $\frac{d\underline{u}}{dt} = \frac{\delta \underline{u}}{\delta t} + \underline{\omega} \times \underline{u}$]

For example (as will be seen below), (9) may be applied with $\underline{u}_F(t)$ as the velocity vector, or as the angular velocity vector $\underline{\omega}_F(t)$.

(E) Equation of motion

(E/1) Consider a particle of mass m at P, subject to a force $\underline{F}(t)$ in the fixed (inertial) frame, so that $\underline{F}(\mathrm{t}) = m\ddot{\underline{r}}_F(t)$,

(noting that Newton's 2nd Law is only valid in an inertial frame).

Then, from (9) with $\underline{u}_F(t)$ as the velocity vector $\dot{\underline{r}}_F$,

$$\underline{F}(\mathrm{t}) = m(\frac{\delta \dot{\underline{r}}_F}{\delta t} + \underline{\omega}_F(t) \times \dot{\underline{r}}_F(t))$$

$$= m\frac{\delta}{\delta t}\left(\frac{\delta \underline{r}_F}{\delta t} + \underline{\omega}_F(t) \times \underline{r}_F(t)\right)$$

$+m\underline{\omega}_F(t) \times \left(\frac{\delta \underline{r}_F}{\delta t} + \underline{\omega}_F(t) \times \underline{r}_F(t)\right)$, from (7) or (7′)

$$= m\frac{\delta^2 \underline{r}_F}{\delta t^2} + m\frac{\delta}{\delta t}\left(\underline{\omega}_F(t) \times \underline{r}_F(t)\right)$$

$$+ m\underline{\omega}_F(t) \times \frac{\delta \underline{r}_F}{\delta t} + m\underline{\omega}_F(t) \times (\underline{\omega}_F(t) \times \underline{r}_F(t))$$

$$= m\frac{\delta^2 \underline{r}_F}{\delta t^2} + m\left(\frac{\delta \underline{\omega}_F}{\delta t} \times \underline{r}_F(t) + \underline{\omega}_F(t) \times \frac{\delta \underline{r}_F}{\delta t}\right)$$

[the above step would need to be justified, for a fully rigorous proof]

$$+ m\underline{\omega}_F(t) \times \frac{\delta \underline{r}_F}{\delta t} + m\underline{\omega}_F(t) \times (\underline{\omega}_F(t) \times \underline{r}_F(t))$$

$$= m\frac{\delta^2 \underline{r}_F}{\delta t^2} + m\frac{\delta \underline{\omega}_F}{\delta t} \times \underline{r}_F(t) + 2m\underline{\omega}_F(t) \times \frac{\delta \underline{r}_F}{\delta t}$$

$$+ m\underline{\omega}_F(t) \times (\underline{\omega}_F(t) \times \underline{r}_F(t))$$

And, from (9) again, with $\underline{\omega}_F(t)$ instead of $\underline{u}_F(t)$:

$$\underline{\dot{\omega}}_F(t) = \frac{\delta \underline{\omega}_F}{\delta t} + \underline{\omega}_F(t) \times \underline{\omega}_F(t) = \frac{\delta \underline{\omega}_F}{\delta t} \quad ,$$

so that $\underline{F}(\mathrm{t}) = m\frac{\delta^2 \underline{r}_F}{\delta t^2} + m\underline{\dot{\omega}}_F(t) \times \underline{r}_F(t) + 2m\underline{\omega}_F(t) \times \frac{\delta \underline{r}_F}{\delta t}$

$$+ m\underline{\omega}_F(t) \times (\underline{\omega}_F(t) \times \underline{r}_F(t))$$

If reference to the fixed frame is assumed throughout, as well as dependence on time, then we can write this more concisely as:

Equation of motion in the fixed frame:

$$\underline{F} = m[\frac{\delta^2 r}{\delta t^2} + \underline{\dot{\omega}} \times \underline{r} + 2\underline{\omega} \times \frac{\delta r}{\delta t} + \underline{\omega} \times (\underline{\omega} \times \underline{r})] \quad (10)$$

(E/2) Newton's 2nd Law can be thought of as applying in the rotating frame, if we rearrange (10) as below:

Equation of motion in the rotating frame:

$$\underline{F} - m\underline{\dot{\omega}} \times \underline{r} - 2m\underline{\omega} \times \frac{\delta r}{\delta t} - m\underline{\omega} \times (\underline{\omega} \times \underline{r}) = m\frac{\delta^2 r}{\delta t^2} \quad (11)$$

considering the extra terms on the lefthand side to be 'apparent' forces

(F) Apparent forces

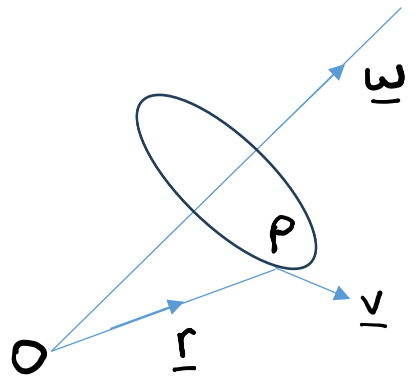
(F/1) The term $-m\underline{\dot{\omega}} \times \underline{r}$ will vanish if $\underline{\omega}$ is constant.

(F/2) $-2m\underline{\omega} \times \frac{\delta r}{\delta t}$ is known as the Coriolis force, and depends on the velocity $\frac{\delta r}{\delta t}$ of the particle relative to the rotating frame.

It is perpendicular to both $\underline{\omega}$ and $\frac{\delta r}{\delta t}$ (by the definition of the vector product); ie it is in the Oxy plane, perpendicular to the particle's path at any instant.

(F/3) $-m\underline{\omega} \times (\underline{\omega} \times \underline{r})$ is the centrifugal force

To determine its direction:



$\underline{v} = \underline{\omega} \times \underline{r}$ (see diagram) is in the plane perpendicular to $\underline{\omega}$,

and so $\underline{\omega} \times (\underline{\omega} \times \underline{r})$ is in this plane (being perpendicular to $\underline{\omega}$) and towards the axis of rotation (being perpendicular to $\underline{\omega} \times \underline{r}$), by the right-hand rule.

[Considering $\underline{c} = \underline{a} \times \underline{b}$, two versions of the right-hand rule are:

(i) Suppose that the plane containing $\underline{a}$ and $\underline{b}$ is a table top, such that, as we look down on the table, an anticlockwise rotation takes us from $\underline{a}$ to $\underline{b}$ (the fingers of the right hand are curled in an anti-clockwise direction). Then $\underline{c}$ (the thumb) will be pointing vertically upwards.

(ii) If the thumb (of your right hand) is pointing in the direction of $\underline{a}$, and the index finger is pointing in the direction of $\underline{b}$, then the middle finger will be pointing in the direction of $\underline{c}$ (assuming that the thumb and fingers are at right-angles to each other).]

Hence $-m\underline{\omega} \times (\underline{\omega} \times \underline{r})$ is in this plane and away from the axis of rotation.

[Note that, by contrast, the centripetal force on a particle moving in a circle is a force on the particle towards the axis of rotation (eg supplied by the tension in a string), that keeps the particle on its circular path.]

(G) Examples

Example 1

An xyz coordinate system is rotating with respect to an XYZ coordinate system, having the same Origin and assumed to be fixed in space (ie it is an inertial system). The angular velocity of the xyz system relative to the XYZ system is given by $\underline{\omega} = 2\underline{i} + t\underline{j} - 3t^2\underline{k}$ (where t is the time in seconds).

The position vector of a particle at time t as observed in the xyz system is given by $\underline{r} = t^3\underline{i} - 2t\underline{j} + \underline{k}$ (in metres)

Find at time $t = 1$: (a) the apparent velocity, and (b) the true velocity.

Solution

[Note: At a particular time t, the xyz axes will be orientated in a certain way relative to the XYZ axes (as determined by the history of $\underline{\omega}$), and so $\underline{\omega}$ at time t could (in principle) be expressed in terms of $\underline{I}$, $\underline{J}$, $\underline{K}$ (the unit vectors associated with the XYZ axes).

So the given information completely specifies the behaviour of the rotating axes relative to the fixed axes; ie it doesn't matter that $\underline{\omega}$ is given in terms of $\underline{i}, \underline{j}, \underline{k}$. Because it is natural for $\underline{r}$ to be given in terms of $\underline{i}, \underline{j}, \underline{k}$ (being the position of the particle relative to the rotating axes), we will in any case want $\underline{\omega}$ to be given in terms of $\underline{i}, \underline{j}, \underline{k}$, in order to be able to determine $\underline{\omega} \times \underline{r}$.]

(a) The apparent velocity at time t is $\frac{\delta \underline{r}}{\delta t} = 3t^2\underline{i} - 2\underline{j}$

At time $t = 1$ this is $3\underline{i} - 2\underline{j}\ ms^{-1}$.

(b) Now, $\underline{\omega} \times \underline{r} = (2\underline{i} + t\underline{j} - 3t^2\underline{k}) \times (t^3\underline{i} - 2t\underline{j} + \underline{k})$

$$= \begin{vmatrix} \underline{i} & 2 & t^3 \\ \underline{j} & t & -2t \\ \underline{k} & -3t^2 & 1 \end{vmatrix} = (t - 6t^3)\underline{i} - (2 + 3t^5)\underline{j} + (-4t - t^4)\underline{k} ,$$

so that the true velocity at time t is $\frac{\delta \underline{r}}{\delta t} + \underline{\omega} \times \underline{r}$

$$= \left(3t^2\underline{i} - 2\underline{j}\right) + ((t - 6t^3)\underline{i} - (2 + 3t^5)\underline{j} + (-4t - t^4)\underline{k})$$

$$= (3t^2 + t - 6t^3)\underline{i} + (-4 - 3t^5)\underline{j} + (-4t - t^4)\underline{k}$$

At time $t = 1$ this is $-2\underline{i} - 7\underline{j} - 5\underline{k}\ \ ms^{-1}$.

Example 2

A bead on a straight wire slides in such a way that its displacement along the wire is $Acos\lambda t$. The wire rotates with constant angular speed ω about an axis which is perpendicular to the wire and passes through the middle of the wire O. Find the velocity and acceleration of the bead relative to the fixed surroundings.

Solution

[This is a planar case.]

Let Ox and Oy be axes along and perpendicular to the wire, and let OX and OY be axes fixed relative to the surroundings, such that the two sets of axes coincide at time $t = 0$.

Method 1 (applying the theory in this note step by step, rather than just the formula in (11))

The position vector of the bead with respect to the fixed frame is $\underline{r}_F(t) = Acos(\lambda t)\underline{i}(t)$, where $\underline{i}(t)$ is a unit vector along the wire, and from (2):

$\underline{i}(t) = cos\theta(t)\underline{I} + sin\theta(t)\underline{J}$ and $\underline{j}(t) = -sin\theta(t)\underline{I} + cos\theta(t)\underline{J}$

The 'velocity relative to the rotating frame' (which is one component of the velocity relative to the fixed frame) is

$\frac{\delta \underline{r}_F(t)}{\delta t} = -A\lambda sin(\lambda t)\underline{i}(t)$

[strictly speaking, the velocity relative to the rotating frame is

$= -A\lambda sin(\lambda t)\underline{i}$, with $\underline{i}$ being fixed]

From (7), $\dot{\underline{r}}_F(t) = \frac{\delta \underline{r}_F}{\delta t} + \underline{\omega}_F(t) \times \underline{r}_F(t)$, and $\underline{\omega}_F(t) = \omega \underline{k}(t)$

so that $\dot{\underline{r}}_F(t) = -A\lambda sin(\lambda t)\underline{i}(t) + \omega \underline{k}(t) \times Acos(\lambda t)\underline{i}(t)$

$= -A\lambda sin(\lambda t)\underline{i}(t) + \omega Acos(\lambda t)\underline{j}(t)$ (*)

Then, from (2) again, with $\theta = \omega t$:

$\dot{\underline{r}}_F(t) = -A\lambda sin(\lambda t)(\cos(\omega t)\underline{I} + sin(\omega t)\underline{J})$

$+ \omega Acos(\lambda t)(-sin(\omega t)\underline{I} + cos(\omega t)\underline{J})$

$= -A(\lambda sin(\lambda t)cos(\omega t) + \omega cos(\lambda t)sin(\omega t))\underline{I}$

$+A(\omega cos(\lambda t)cos(\omega t) - \lambda sin(\lambda t)sin(\omega t))\underline{J}$

The acceleration relative to the fixed frame can be found from

(9): $\dot{\underline{u}}_F(t) = \frac{\delta \underline{u}_F(t)}{\delta t} + \underline{\omega}_F(t) \times \underline{u}_F(t)$, with $\underline{u}_F(t) = \dot{\underline{r}}_F(t)$,

so that $\ddot{\underline{r}}_F(t) = \frac{\delta \dot{\underline{r}}_F}{\delta t} + \underline{\omega}_F(t) \times \dot{\underline{r}}_F(t)$

From (*), $\dot{\underline{r}}_F(t) = -A\lambda sin(\lambda t)\underline{i}(t) + \omega Acos(\lambda t)\underline{j}(t)$,

so that $\frac{\delta \dot{\underline{r}}_F}{\delta t} = -A\lambda^2 cos(\lambda t)\underline{i}(t) - \lambda\omega Asin(\lambda t)\underline{j}(t)$

[treating $\underline{i}(t)$ and $\underline{j}(t)$ as constant, for the purposes of $\frac{\delta}{\delta t}$]

And $\underline{\omega}_F(t) \times \underline{\dot{r}}_F(t) = \omega\underline{k}(t) \times (-A\lambda sin(\lambda t)\underline{i}(t) + \omega Acos(\lambda t)\underline{j}(t))$

$= -\omega A\lambda sin(\lambda t)\underline{j}(t) - \omega^2 Acos(\lambda t)\underline{i}(t)$

So $\underline{\ddot{r}}_F(t) = \frac{\delta \underline{\dot{r}}_F}{\delta t} + \underline{\omega}_F(t) \times \underline{\dot{r}}_F(t)$

$= \left[-A\lambda^2 cos(\lambda t)\underline{i}(t) - \lambda\omega Asin(\lambda t)\underline{j}(t)\right]$

$+[-\omega A\lambda sin(\lambda t)\underline{j}(t) - \omega^2 Acos(\lambda t)\underline{i}\,(t)]$

$= -A(\lambda^2 + \omega^2)\cos(\lambda t)\,\underline{i}(t) - 2A\omega\lambda sin(\lambda t)\underline{j}(t)$

Then, as $\underline{i}(t) = cos\theta(t)\underline{I} + sin\theta(t)\underline{J}$

and $\underline{j}(t) = -sin\theta(t)\underline{I} + cos\theta(t)\underline{J}$, and $\theta = \omega t$:

$\underline{\ddot{r}}_F(t) = -A(\lambda^2 + \omega^2)\cos(\lambda t)(\cos\,(\omega t)\underline{I} + \sin\,(\omega t)\underline{J})$

$-2A\omega\lambda sin(\lambda t)(-sin(\omega t)\underline{I} + cos(\omega t)\underline{J})$

$= [-A(\lambda^2 + \omega^2)\cos(\lambda t)\,cos(\omega t) + 2A\omega\lambda sin(\lambda t)sin(\omega t)]\underline{I}$

$+[-A(\lambda^2 + \omega^2)\cos(\lambda t)\,sin(\omega t) - 2A\omega\lambda sin(\lambda t)cos(\omega t)]\underline{J}$

Method 2 (independent of the above theory)

$\underline{r}_F(t) = Acos(\lambda t)\underline{i}(t)$ and $\underline{i}(t) = cos\theta(t)\underline{I} + sin\theta(t)\underline{J}$,

so that $\underline{r}_F(t) = Acos(\lambda t)(cos\theta(t)\underline{I} + sin\theta(t)\underline{J})$

Then, as $\theta = \omega t$,

$\underline{r}_F(t) = Acos(\lambda t)\cos(\omega t)\,\underline{I} + Acos(\lambda t)\sin(\omega t)\,\underline{J}$

Thus, writing $\underline{r}_F(t) = X(t)\underline{I} + Y(t)\underline{J}$,

$X(t) = Acos(\lambda t)\cos(\omega t)$ and $Y(t) = Acos(\lambda t)\sin(\omega t)$

Hence $\dot{X}(t) = -A\lambda\sin(\lambda t)\cos(\omega t) - Acos(\lambda t).\,\omega\sin(\omega t)$

and $\dot{Y}(t) = -A\lambda\sin(\lambda t)\sin(\omega t) + Acos(\lambda t).\,\omega cos(\omega t)$

Then $\underline{\dot{r}}_F(t) = \dot{X}(t)\underline{I} + \dot{Y}(t)\underline{J}$

$= -A(\lambda sin(\lambda t)cos(\omega t) + \omega cos(\lambda t)sin(\omega t))\underline{I}$

$+A(\omega cos(\lambda t)cos(\omega t) - \lambda sin(\lambda t)sin(\omega t))\underline{J}$,

and $\underline{\ddot{r}}_F(t) = \ddot{X}(t)\underline{I} + \ddot{Y}(t)\underline{J}$,

where $\ddot{X}(t) = -A\lambda^2\cos(\lambda t)\cos(\omega t) + A\lambda\omega\sin(\lambda t)\sin(\omega t)$

$+A\lambda\omega sin(\lambda t)\sin(\omega t) - A\omega^2 cos(\lambda t)\cos(\omega t)$

and $\ddot{Y}(t) = -A\lambda^2 cos(\lambda t)\sin(\omega t) - A\lambda\omega\sin(\lambda t)\cos(\omega t)$

$- A\lambda\omega sin(\lambda t).\cos(\omega t) - A\omega^2 cos(\lambda t)sin(\omega t)$,

so that

$\underline{\ddot{r}}_F(t) = [-A(\lambda^2 + \omega^2)\cos(\lambda t)\,cos(\omega t) + 2A\omega\lambda sin(\lambda t)sin(\omega t)]\underline{I}$

$+[-A(\lambda^2 + \omega^2)\cos(\lambda t)\,sin(\omega t) - 2A\omega\lambda sin(\lambda t)cos(\omega t)]\underline{J}$

Example 3

A straight wire of length $2a$ rotates with constant angular velocity ω about a fixed perpendicular axis through the centre of the wire. A bead of mass m is placed on the wire at its mid-point and released. If friction is negligible and the wire rotates in a horizontal plane, establish a differential equation for the motion of the bead, relative to the wire.

Solution

[This is a planar case again.]

Once again, let Ox and Oy be axes along and perpendicular to the wire, and let OX and OY be axes fixed relative to the surroundings, such that the two sets of axes coincide at time $t = 0$.

From (11), the equation of motion for the bead, relative to the wire is $\underline{F} - m\underline{\dot{\omega}} \times \underline{r} - 2m\underline{\omega} \times \frac{\delta \underline{r}}{\delta t} - m\underline{\omega} \times (\underline{\omega} \times \underline{r}) = m\frac{\delta^2 \underline{r}}{\delta t^2}$

with $\underline{F} = R\underline{j}$, $\underline{\omega} = \omega\underline{k}$, $\underline{r} = x\underline{i}$

so that $\underline{\dot{\omega}} = \underline{0}$, $\frac{\delta \underline{r}}{\delta t} = \dot{x}\underline{i}$, $\frac{\delta^2 \underline{r}}{\delta t^2} = \ddot{x}\underline{i}$

Hence $R\underline{j} - 2m\omega\underline{k} \times \dot{x}\underline{i} - m\omega\underline{k} \times (\omega\underline{k} \times x\underline{i}) = m\ddot{x}\underline{i}$,

so that $R\underline{j} - 2m\omega\dot{x}\underline{j} - m\omega\underline{k} \times \left(\omega x\underline{j}\right) = m\ddot{x}\underline{i}$,

and $R\underline{j} - 2m\omega\dot{x}\underline{j} + m\omega^2 x\underline{i} = m\ddot{x}\underline{i}$

Then, equating $\underline{i}$ terms: $\ddot{x} = \omega^2 x$ (the equation of motion of the bead), (and also, equating $\underline{j}$ terms: $R = 2m\omega\dot{x}$).

Appendix: List of results

(B/2)(ii) $\underline{\dot{r}}_F(t) = \dot{x}(t)\underline{i}(t) + \dot{y}(t)\underline{j}(t) + \dot{z}(t)\underline{k}(t)$

$+ x(t)\frac{d}{dt}\underline{i}(t) + y(t)\frac{d}{dt}\underline{j}(t) + z(t)\frac{d}{dt}\underline{k}(\text{t})$ (1)

(C1/1)

$\underline{i}(t) = cos\theta(t)\underline{I} + sin\theta(t)\underline{J}$ and $\underline{j}(t) = -sin\theta(t)\underline{I} + cos\theta(t)\underline{J}$ (2)

(C1/4) $\frac{d}{dt}\underline{i}(t) = \omega(t)\underline{j}(t)$; $\frac{d}{dt}\underline{j}(t) = -\omega(t)\underline{i}(t)$ (3)

(C2/2) $\frac{d\underline{i}(t)}{dt} = \underline{\omega}_F(t) \times \underline{i}(t)$, $\frac{d\underline{j}(t)}{dt} = \underline{\omega}_F(t) \times \underline{j}(t)$

and $\frac{d\underline{k}(t)}{dt} = \underline{\omega}_F(t) \times \underline{k}(t)$ (4)

(D1/1)

$\underline{\dot{r}}_F(t) = \dot{x}(t)\underline{i}(t) + \dot{y}(t)\underline{j}(t) + x(t)\omega(t)\,\underline{j}(t) - y(t)\omega(t)\,\underline{i}(t)$ (5)

(D1/2) $\underline{\dot{r}}_F(t) = \dot{x}(t)\underline{i}(t) + \dot{y}(t)\underline{j}(t) + \underline{\omega}_F(t) \times \underline{r}_F(t)$ (6)

(D1/2) $\underline{\dot{r}}_F(t) = \frac{\delta \underline{r}_F}{\delta t} + \underline{\omega}_F(t) \times \underline{r}_F(t)$ (7)

$\frac{d\underline{r}}{dt} = \frac{\delta \underline{r}}{\delta t} + \underline{\omega} \times \underline{r}$ (8)

(D2/1) $\underline{\dot{r}}_F(t) = \frac{\delta \underline{r}_F}{\delta t} + \underline{\omega}_F(t) \times \underline{r}_F(t)$ (7′)

(D2/1) $\frac{d\underline{r}}{dt} = \frac{\delta \underline{r}}{\delta t} + \underline{\omega} \times \underline{r}$ (8′)

(D3) $\underline{\dot{u}}_F(t) = \frac{\delta \underline{u}_F(t)}{\delta t} + \underline{\omega}_F(t) \times \underline{u}_F(t)$ (9)

(E/1) $\underline{F} = m[\frac{\delta^2 \underline{r}}{\delta t^2} + \underline{\dot{\omega}} \times \underline{r} + 2\underline{\omega} \times \frac{\delta \underline{r}}{\delta t} + \underline{\omega} \times (\underline{\omega} \times \underline{r})]$ (10)

(E/2) $\underline{F} - m\underline{\dot{\omega}} \times \underline{r} - 2m\underline{\omega} \times \frac{\delta \underline{r}}{\delta t} - m\underline{\omega} \times (\underline{\omega} \times \underline{r}) = m\frac{\delta^2 \underline{r}}{\delta t^2}$ (11)