Rotating Bodies (5 pages; 19/8/19)

See also "Rotating Bodies - Examples" and "Moments of Inertia".

This note concerns situations involving an object that rotates about a fixed axis. There may also be other objects involved: a typical system would be a particle sliding on a rotating rod.

(A) Overview

(1) The simplest situation that we deal with in Mechanics is that of a particle (or assumed particle). In this case, we can draw a force diagram and apply Newton's 2nd Law in two perpendicular directions. Although its familiar form is $F = ma$, the 2nd Law can also be expressed as: $F = \frac{dp}{dt}$, where $p = mv$;

ie force = rate of change of (linear) momentum

(we are assuming that the mass is constant; ie a non-relativistic model).

Alternative methods at our disposal are:

(a) conservation of (linear) momentum (eg two balls colliding), or

(b) conservation of energy (eg an object moving in a vertical circle) or (as the more generalised form) the work-energy principle; ie work done = increase in kinetic energy

Note: Conservation of (linear) momentum is the special case of $F = \frac{dp}{dt}$ when $F = 0$.

(2) The next level of complexity is to allow the object to have dimensions (ie it is no longer being treated as a particle). If an

object is in ('linear') equilibrium, it can be shown that if we take moments about any point, then we always get the same value.

If we restrict ourselves to situations where the object is in ('linear') equilibrium and does not rotate, then:

total moment of forces (about any point) = 0

[The object is in rotational equilibrium. This would also apply to cases where the object was rotating at a constant rate (just as an object that moves with constant (linear) velocity is said to be in equilibrium).]

(3) Beyond these simple situations, it is possible for an object to be accelerating linearly and also rotating. This is the most general situation and is covered at university level.

A halfway house is where there is no linear motion, and the object is rotating about a fixed axis (obviously this is quite a restriction!)

However, most STEP problems will involve more than one object - in a similar way to having balls colliding in linear motion, where we can consider the system as a whole when looking at conservation of momentum.

The approach to solving problems will be to apply the rotational equivalents of the methods used for linear motion.

So, in place of

"(total) force = rate of change of (linear) momentum", we have

"(total) moment of forces = rate of change of angular momentum"

[special case, when there are no external forces: in place of

"conservation of (linear) momentum" we have "conservation of angular momentum"]

and the work-energy principle can be applied, taking account of the (rotational) work done by the moments of forces and the rotational kinetic energy [special case, when there are no external forces: conservation of energy]

Before we can apply these methods, we will need to consider the concepts of angular velocity and moment of inertia.

(B) Angular Velocity

Consider a particle moving in a circular path, with velocity v (the term 'angular speed' is often used instead, though the direction of rotation would have to be specified).

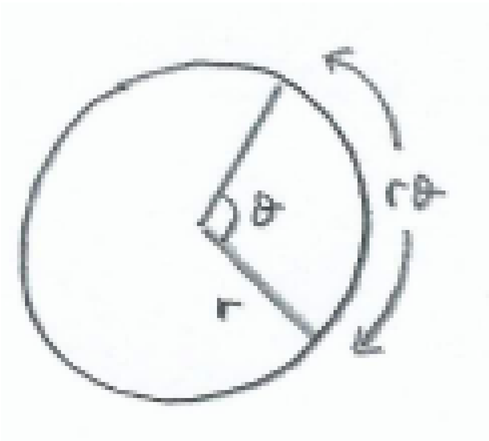

Figure 1

Referring to Figure 1, arc length $s = r\theta$

The angular velocity is $\frac{d\theta}{dt}$ or $\dot{\theta}$

And $v = \frac{ds}{dt} = \frac{d}{dt}(r\theta) = r\frac{d\theta}{dt} = r\dot{\theta}$

[Note: When $\dot{\theta}$ is constant it is often given the symbol ω.]

(C) Moment of Inertia - see separate note

(D) Methods available for tackling rotating body problems

(1) Summary of correspondence between linear and angular quantities (see below for further details)

Linear	Angular	Notes
s	θ	
v	$\dot{\theta}$	
a	$\ddot{\theta}$	
m	I	I is the moment of inertia
mv	$I\dot{\theta}$	$I\dot{\theta}$ is angular momentum (often denoted by L)
$\frac{1}{2}mv^2$	$\frac{1}{2}I\left(\dot{\theta}\right)^2$	kinetic energy
$F = ma$	$C = I\ddot{\theta}$	C is moment of force (aka couple or torque)
$\int F\, ds$	$\int C\, d\theta$	work done
$\int F\, dt = [mv]$	$\int C\, dt = [I\dot{\theta}]$	$\int C\, dt$ is the impulse of a torque; the square brackets denote "change in"

Note: The term 'couple' implies two equal but opposite forces applied to an object, causing it to rotate (without any translation). But it is sometimes used to represent the combined effect of any number of forces on an object that is rotating; ie it is the total moment of the forces. (Such a system can be reduced to an equivalent two force situation.)

(2) Total moment of forces about the axis of rotation

= rate of change of angular momentum: $\sum C = I\ddot{\theta}$

[Special case, where there are no external forces: conservation of angular momentum]

Note that the angular momentum of a particle is:

$I\dot{\theta} = (mr^2)\left(\frac{v}{r}\right) = (mv)r$ ("moment of momentum" is an alternative term for angular momentum)

(3) Work-Energy equation: $\int C\, d\theta = \left[\frac{1}{2}I\left(\dot{\theta}\right)^2\right]$

(where the square brackets represent "change in")

ie total work done by moments = change in rotational kinetic energy

Derivation: $\int C\, d\theta = \int I\ddot{\theta}\, d\theta = \int I\ddot{\theta}\frac{d\theta}{dt}dt = \int I\ddot{\theta}\dot{\theta}dt$

$= \left[\frac{1}{2}I\left(\dot{\theta}\right)^2\right]$ (since $\frac{d}{dt}\left(\frac{1}{2}I\left(\dot{\theta}\right)^2\right) = I\dot{\theta}\ddot{\theta}$)

[Special case, where there are no external forces: conservation of energy]

(4) Impulses

Integrating $C = I\ddot{\theta}$ (where C now denotes the total moment of forces (or torque)), we obtain:

$\int C\, dt = [I\dot{\theta}]$

In words: the total impulse of a torque about a given axis of rotation equals the change in angular momentum of the body about the same axis.