Roots of Polynomial Equations - Exercises (Solutions)

(6 pages; 21/3/20)

(1**) If the quadratic equation $2x^2 + 5x - 9 = 0$ has roots α and β, find the quadratic equation which has roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$

Solution

Method 1

$\alpha + \beta = -\frac{5}{2}$ and $\alpha\beta = -\frac{9}{2}$

Let the new equation be $x^2 + bx + c = 0$

Then $\frac{1}{\alpha} + \frac{1}{\beta} = -b$ and $\frac{1}{\alpha} . \frac{1}{\beta} = c$,

so that $b = \frac{-(\alpha+\beta)}{\alpha\beta} = -\frac{5}{9}$ and $c = -\frac{2}{9}$

and the new equation is $x^2 - \frac{5x}{9} - \frac{2}{9} = 0$

or $9x^2 - 5x - 2 = 0$

[Note that, if written as $-9x^2 + 5x + 2 = 0$, then the coefficients of the original equation are reversed.]

Method 2

Let $u = \frac{1}{x}$, so that $x = \frac{1}{u}$

Then $2\left(\frac{1}{u}\right)^2 + \frac{5}{u} - 9 = 0$

and $2 + 5u - 9u^2 = 0$ or $9u^2 - 5u - 2 = 0$

(2***) If the roots of the equation $x^2 + x - 13 = 0$ are α & β, find the equation with roots $2\alpha + 3\beta$ & $3\alpha + 2\beta$

Solution

Let the new equation be $x^2 + bx + c = 0$

Then $-b = (2\alpha + 3\beta + 3\alpha + 2\beta) = 5(\alpha + \beta) = 5(-1)$

And $c = (2\alpha + 3\beta)(3\alpha + 2\beta) = 6(\alpha^2 + \beta^2) + 13\alpha\beta$

$= 6\{(\alpha + \beta)^2 - 2\alpha\beta\} + 13\alpha\beta = 6(\alpha + \beta)^2 + \alpha\beta$

$= 6(-1)^2 - 13 = -7$

Hence the new equation is $x^2 + 5x - 7 = 0$

(3**) If the roots of the equation $x^3 - 14x^2 + 56x - 64 = 0$ are $\alpha,\ \beta\ \&\ \gamma$, find the equation with roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ & $\frac{1}{\gamma}$

Solution

Substitution method: Let $u = \frac{1}{x}$, so that $x = \frac{1}{u}$

Then $\left(\frac{1}{u}\right)^3 - 14\left(\frac{1}{u}\right)^2 + 56\left(\frac{1}{u}\right) - 64 = 0$

and $1 - 14u + 56u^2 - 64u^3 = 0$

or $64u^3 - 56u^2 + 14u - 1 = 0$ (coefficients are reversed)

(4***) Find the roots of the equation $x^3 - 14x^2 + 56x - 64 = 0$,

given that they form a geometric progression.

Solution

Let the roots be $\frac{\alpha}{r}$, α & $r\alpha$

Then $\frac{\alpha}{r}$. α . $r\alpha = 64$, so that $\alpha = 4$

Also $\frac{\alpha}{r} + \alpha + r\alpha = 14$, so that $\frac{1}{r} + 1 + r = \frac{7}{2}$

Then $2(1 + r + r^2) = 7r$ and $2r^2 - 5r + 2 = 0$

Hence $(2r - 1)(r - 2) = 0$ and so $r = \frac{1}{2}$ or 2

Thus the roots are 2, 4 and 8.

(5***) If the roots of the equation $x^5 + bx^4 + cx^3 + \mathrm{d}x^2 + ex + f = 0$ are 5 consecutive positive integers, find expressions for these roots.

Solution

Let the roots be $\alpha - 2$, $\alpha - 1$, α, $\alpha + 1$ & $\alpha + 2$

Then, summing these, $5\alpha = -b$

and hence the roots are $-(\frac{b}{5} + 2)$, $-(\frac{b}{5} + 1)$, $-\frac{b}{5}$, $1 - \frac{b}{5}$ & $2 - \frac{b}{5}$

(6***) If α, β and γ are the roots of the equation

$x^3 - 14x^2 + 56x - 64 = 0$,

find the equation with roots $\alpha\beta$, $\alpha\gamma$ and $\beta\gamma$.

Solution

Let u$= \alpha\beta = \frac{\alpha\beta\gamma}{\gamma} = \frac{64}{\gamma}$

Then $\gamma = \frac{64}{u}$ satisfies the original equation

Similarly for $u = \alpha\gamma$ and $u = \beta\gamma$.

Thus the required equation is

$$\left(\frac{64}{u}\right)^3 - 14\left(\frac{64}{u}\right)^2 + 56\left(\frac{64}{u}\right) - 64 = 0,$$

giving $4096 - 896u + 56u^2 - u^3 = 0$

or $u^3 - 56u^2 + 896u - 4096 = 0$

(7***) If α, β and γ are the roots of the equation

$x^3 - 2x^2 - 4x + 5 = 0$,

find the equation with roots $\alpha + \beta\gamma$, $\beta + \alpha\gamma$ and $\gamma + \alpha\beta$.

Solution

Let the new equation be $x^3 + bx^2 + cx + d = 0$

Then $b = -(\alpha + \beta\gamma + \beta + \alpha\gamma + \gamma + \alpha\beta)$

$= -\sum\alpha - \sum\alpha\beta = -2 - (-4) = 2$

$c = (\alpha + \beta\gamma)(\beta + \alpha\gamma) + (\alpha + \beta\gamma)(\gamma + \alpha\beta) + (\beta + \alpha\gamma)(\gamma + \alpha\beta)$

$= (\alpha\beta + \alpha^2\gamma + \beta^2\gamma + \alpha\beta\gamma^2) +$...

[By symmetry, this contains all the types of terms appearing in the full expansion, and there are $3(4) = 12$ terms.]

$= \sum \alpha\beta + \sum \alpha^2\beta + \sum \alpha\beta\gamma^2$

[As a check, this contains $3 + 6 + 3 = 12$ terms]

Thus $c = (-4) + \sum \alpha^2\beta + \alpha\beta\gamma \sum \alpha$

$(-4) + \sum \alpha^2\beta + (-5)(2) = -14 + \sum \alpha^2\beta$ (A)

[$\sum \alpha^2\beta$ to be found shortly]

And $d = -(\alpha + \beta\gamma)\big(\beta + \alpha\gamma\big)(\gamma + \alpha\beta)$

[this will give $2^3 = 8$ terms]

$= -(\alpha\beta\gamma + (\sum \alpha^2\beta^2) + \alpha^2\beta^2\gamma^2 + \sum \alpha^3\beta\gamma\ )$

[This can be obtained by performing the expansion, but only noting the types of term (some of which are repeated).]

[$1 + 3 + 1 + 3 = 8$ terms]

Thus $d = -(-5) - \sum \alpha^2\beta^2 - (-5)^2 - \alpha\beta\gamma \sum \alpha^2$

$= -20 - \sum \alpha^2\beta^2 - (-5) \sum \alpha^2$ (B)

So we need to find $\sum \alpha^2$, $\sum \alpha^2\beta^2$ & $\sum \alpha^2\beta$

First of all, consider $(\alpha + \beta + \gamma)^2 = \sum \alpha^2 + 2 \sum \alpha\beta$,

so that $\sum \alpha^2 = 2^2 - 2(-4) = 12$

We can also consider $(\alpha\beta + \alpha\gamma + \beta\gamma)^2 = \sum(\alpha\beta)^2 + 2 \sum \alpha^2\beta\gamma$

[giving $3 + 2(3) = 9$ terms]

so that $\sum \alpha^2\beta^2 = (-4)^2 - 2\alpha\beta\gamma \sum \alpha = 16 - 2(-5)(2) = 36$

Then $(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma)=(\sum\alpha^2\beta)+3\alpha\beta\gamma$

[3(2) + 3 = 9 terms]

so that $\sum\alpha^2\beta=2(-4)-3(-5)=7$

Hence, from (A), $c=-14+\sum\alpha^2\beta=-14+7=-7$

and, from (B),

$d=-20-\sum\alpha^2\beta^2-(-5)\sum\alpha^2=-20-36+5(12)=4$

And so the required equation is $x^3+2x^2-7x+4=0$

[In this example we can use the Factor theorem to see that $\alpha\ (say)=1$, and that $\beta,\gamma=\frac{1\pm\sqrt{21}}{2}$, which leads to $\alpha+\beta\gamma$ etc being -4, 1 & 1, enabling the new equation to be confirmed. In general of course, we may not be able to find a root by the Factor theorem.]