Rolling Wheels (12 pages; 14/11/18)

See also STEP 2018, P2, Q11

Contents

(A) Speed relative to the ground - Part 1

Assume that a light hoop (ie of negligible mass) is rolling (without slipping) at a constant speed v on horizontal ground. How fast is the point P on the hoop (in Figure 1) moving relative to the ground (a) when it is touching the ground, and (b) when it is at the top?

Suppose that the hoop is rotating in a clockwise sense and has radius a. Let θ be as in the diagram.

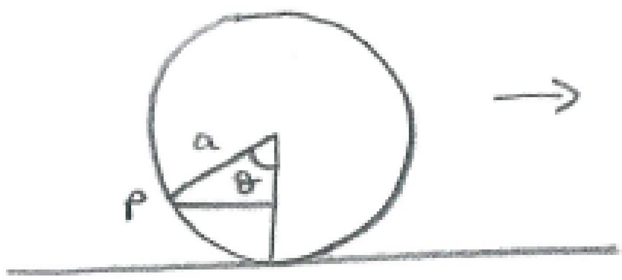


Figure 1

As P leaves the ground, it moves round to the left, and has speed a$\dot{\theta}$ along the circumference of the hoop, relative to the centre of the hoop. [The arc length travelled as the angle increases by θ rad is $a\theta$, and the tangential speed is $\frac{d}{dt}(a\theta) = a\dot{\theta}$ (where $\dot{\theta} \equiv \frac{d\theta}{dt}$)]
The ground that the hoop covers is equal to the distance that P moves along the circumference, and hence the centre of the hoop is also moving with speed a$\dot{\theta}$ (but to the right, and relative to the ground), so that $v = a\dot{\theta}$.

The motion of P relative to the ground has two components: its motion relative to the centre of the hoop, and the motion of the centre of the hoop relative to the ground. When P is touching the ground, its speed relative to the ground is therefore

$-a\dot{\theta} + a\dot{\theta} = 0$ (if motion to the right is considered to be positive); ie P is stationary!

At the top, P has speed $a\dot{\theta} + a\dot{\theta} = 2a\dot{\theta} = v$, relative to the ground.

The fact that the hoop is rolling rather than slipping means that it is effectively toppling continually about the point of contact with the ground. (Consider a circular star-like structure, with a large number of limbs.)

(B) Speed relative to the ground - Part 2

For the same hoop, how fast is the general point P (in Figure 2) moving relative to the ground – in terms of θ? And (separately) in terms of h (the height above the ground)?

Let this speed be v_P (in the direction indicated in Figure 2).

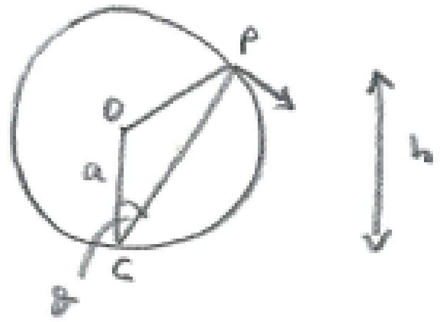


Figure 2

We can use the fact that the point of contact of the hoop with the ground is the 'instantaneous centre of rotation' of the hoop: every point on the hoop is (instantaneously) rotating about C with the same angular speed, ω say.

Every point is therefore instantaneously moving at right angles to the line joining it to C; PC in the case of the point P.

The point at the top of the hoop has speed $2v$ relative to the ground (from Part 1), so that $\omega = \frac{2v}{2a} = \frac{v}{a}$ (1)

[since, in general, the tangential speed v and the angular speed ω are related by $v = \omega r$, where r is the radius of the circular path being followed]

For P, $\omega = \frac{v_P}{PC}$ (2)

As the triangle OPC is isosceles , $PC = 2acos\theta$

Hence, from (1) & (2), $v_P = \frac{v}{a}PC = 2vcos\theta$

Also, $h = PCcos\theta$ (drawing a line from P to the ground, parallel to OC) $= 2acos^2\theta$, so that $v_P = 2v\sqrt{\frac{h}{2a}} = v\sqrt{\frac{2h}{a}}$

Note that both results agree with Part 1 when P is at the top or bottom of the hoop.

Note also that, in Part 1, the angular speed of P about the centre O was $\omega = \frac{v}{a} = \dot{\theta}$; ie the angular speed about the instantaneous centre of rotation is the same as the angular speed about the centre.

(C) Speed relative to the ground - Part 3

Alternative method:

The velocity of point P has two components - translational and rotational.

The translational component is horizontal and equal to v (the same as the centre of the hoop).

The rotational component is perpendicular to OP and is equal to

$a\dot{\theta} = v$ (also).

This immediately gives the speed relative to the ground when P is at the top or bottom.

For general P (referring to Figure 3):

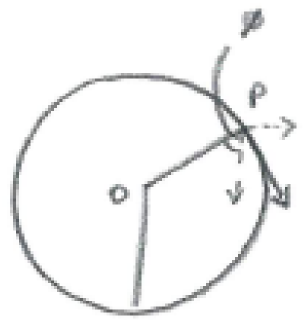


Figure 3

The rotational component can itself be resolved into horizontal and vertical components, as follows:

$\phi = \pi - \angle COP = \pi - (\pi - 2\theta) = 2\theta$

(referring to the isosceles triangle (and the point C) in Figure 2)

Then its vertical component $= vcos\left(\frac{\pi}{2} - \phi\right) = vsin\phi = vsin(2\theta)$

and its horizontal component $= vcos(2\theta)$

Adding in the translational component, gives:

total horizontal component $= v(1 + cos2\theta)$

and total vertical component $= vsin2\theta$,

so that the magnitude of the velocity of P is:

$v\sqrt{(1 + cos2\theta)^2 + sin^2 2\theta}$

$= v\sqrt{(1 + cos^2\theta - \ sin^2\theta)^2 + 4sin^2\theta cos^2\theta}$

$= v\sqrt{(2cos^2\theta)^2 + 4sin^2\theta cos^2\theta}$

$= v\sqrt{4cos^4\theta + 4sin^2\theta cos^2\theta}$

$= v\sqrt{4cos^2\theta(cos^2\theta + \ sin^2\theta)}$

$= 2vcos\theta$, as before

(D) Accelerating wheel

Consider a stationary car that is suddenly accelerated. If the acceleration is not too great and the road surface is sufficiently rough, then friction will be able to prevent the motion of the part of the wheel that is in contact with the road. This causes rolling, with the effect that an adjacent part of the wheel now becomes in contact with the road. As seen in (A), the part of the wheel that is in contact with the road is momentarily at rest relative to the ground (the net effect of the motion forwards of the car and the opposite rotation of the wheel). Thus the friction involved here is static friction (with coefficient μ_s).

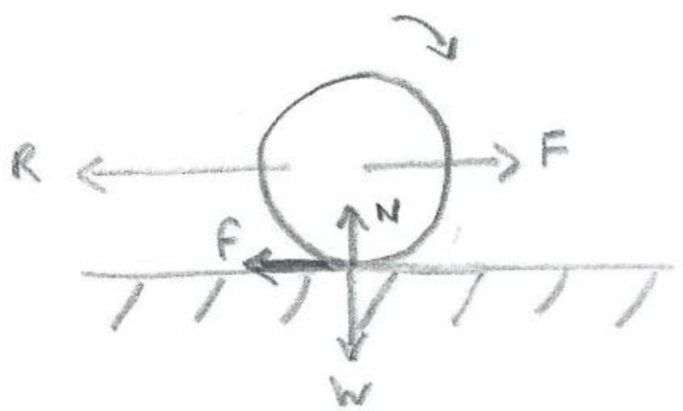
If there is any skidding of the wheels (ie if the acceleration or braking is too great, or if the surface is not rough enough), then there will be dynamic (or kinetic) friction at the point of contact (with coefficient μ_k), instead of static friction (ie as if a block is sliding along a surface).

The direction of friction depends on whether the car is accelerating or decelerating, and (surprisingly perhaps) whether it is being pushed or moves as a result of the engine turning the axle.

Situation A: Force through the centre of mass causing acceleration (eg a car being pushed, or a horse pulling a cart)

[wheel moving to the right]

Figure 4



The frictional force is towards the left because it opposes the attempted motion towards the right (consider the situation where the wheel is initially stationary, and then pushed; ie the situation is similar to a block being pushed on a table).

The net forwards force is $F - R - f$. The frictional force f can be thought of as converting linear acceleration into angular acceleration. No energy is lost due to this friction, because the point of contact is stationary (and therefore no work is done).

Resistances to motion (totalling R in the previous diagram):

(a) air resistance

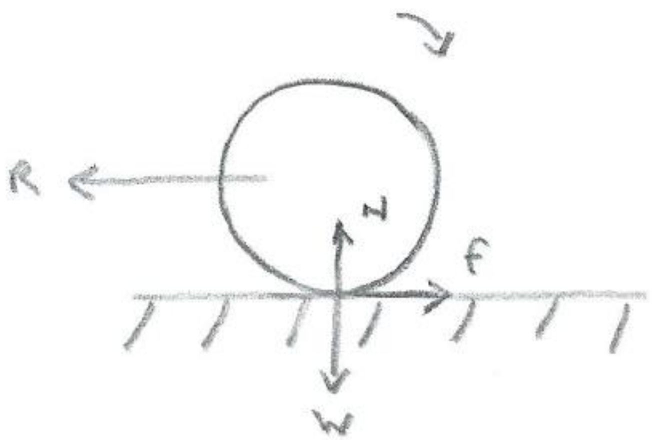
(b) friction at the axle

(c) due to the way that the tyre compresses - this is often referred to as 'rolling friction' (equivalent to μ of about 0.01) – it isn't really friction though (a torque is created, producing a negative angular acceleration).

Note that the size of the equivalent coefficient of friction is very small compared with the usual values; hence the advantage of rolling wheels as a means of transport!

Situation A is revisited, after situation D, once further theory has been covered.

Situation B: Car accelerates via a clockwise torque (ie axle turning due to a force applied by the engine)



[wheel moving to the right]

Consider the part of the wheel that is in contact with the ground. This experiences a force to the left, as the axle rotates.

Friction opposes this, and is therefore to the right, preventing the contact point from moving, and hence causes rolling. (Consider a person walking across ice, where the feet have a tendency to slip backwards (ie to the left if moving to the right), if friction is overcome; so friction acts to the right.)

The acceleration of the wheel has two components: translational and rotational. The translational acceleration is due to the combined effect of friction and the other resistance forces; ie not the driving force of the engine - although this gives rise to the friction.

If the rotational accelerating force exceeds the limiting frictional force, then the wheel slips, with the contact point moving to the left.

Example

With $\mu_s = 0.6$, and ignoring resistance to motion:

The forwards force on the car is the total frictional force.

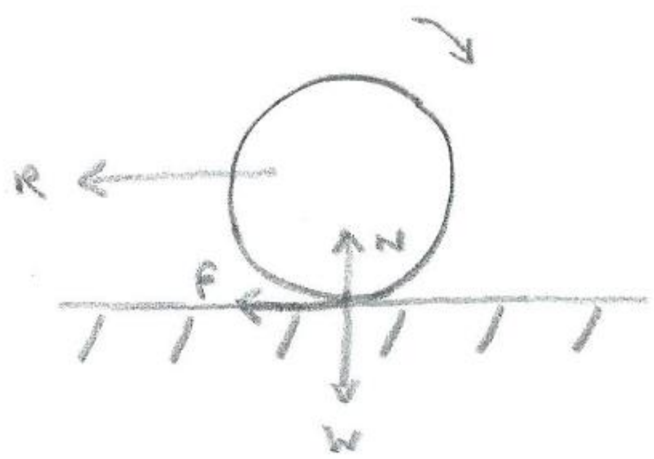
The maximum frictional force per wheel is $\mu\left(\frac{mg}{4}\right)$, and the total maximum frictional force for a rear-wheel drive car is therefore $2\mu\left(\frac{mg}{4}\right)$

Then $2\mu\left(\frac{mg}{4}\right) = ma_{max}$,

and the maximum acceleration of the car is

$\frac{\mu g}{2} = \frac{0.6(9.8)}{2} = 2.94ms^{-1}$

Situation C: Car brakes via an anti-clockwise torque (using brake pads)



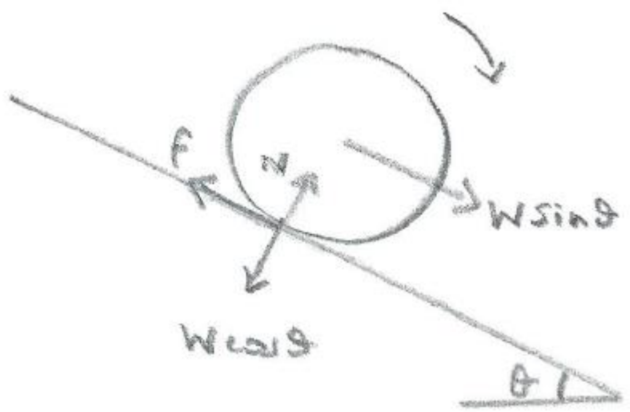
This tends to make the part of the wheel that is in contact with the ground move to the right. Hence friction acts to the left (as it opposes attempted motion).

If limiting friction is exceeded, then there is some skidding - with the contact point moving to the right relative to the ground (though rotation can continue as well).

Where the car brakes without skidding, wheel rotation can be slowed down in a controlled manner, dependent on the coefficient of static friction; whereas with skidding the deceleration depends on the coefficient of dynamic friction.

For a car on a wet surface, the dynamic coefficient of friction, μ_k may be 0.4, compared to the static coefficient of friction, μ_s of perhaps 0.7. Hence, a braking car slows down more rapidly when rolling, rather than skidding.

Situation D: Wheel rolling down a slope



This is essentially the same as Situation A: instead of the wheel being pushed (or pulled), it is subject to the component of the weight down the slope ($Wsin\theta$). As in Situation A, the friction opposes the attempted motion, which is down the slope. So friction acts up the slope.

However, it may be the case that the wheel slips instead of rolling.

Consider the angular equivalent of Newton's 2nd law:

$\tau = I\alpha$,

where τ (torque) $= fr$, and r is the radius of the wheel;

I(moment of inertia of the wheel) $= \frac{1}{2}mr^2$, and m is the mass of the wheel;

α (angular acceleration) $= \frac{a}{r}$, and a is the linear acceleration of the wheel down the slope [in the same way that $v = a\dot{\theta}$ in (A), so that $\dot{\theta} = \frac{v}{a}$]

So, if rolling takes place, $fr = \frac{1}{2}mr^2\left(\frac{a}{r}\right) \Rightarrow f = \frac{1}{2}ma$ (1)

Also, resolving forces down the slope (with $W = mg$),

$mgsin\theta - f = ma$ (2)

and $f \leq \mu mgcos\theta$ (3)

Thus, if rolling takes place, (1) & (2) give:

$mgsin\theta - f = m(\frac{2f}{m})$

$\Rightarrow f = \frac{1}{3}mgsin\theta$

and (3) $\Rightarrow \frac{1}{3}mgsin\theta \leq \mu mgcos\theta \Rightarrow \mu \geq \frac{1}{3}tan\theta$

For example, when $\theta = 30°, \mu \geq \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right) = 0.192$ (3sf)

Also from (1) & (2), $mgsin\theta - \frac{1}{2}ma \;\; = ma$,

so that $a = \frac{2}{3}gsin\theta$ (when there is rolling).

Situation A revisited, for a single ball being pushed along

As in situation D, $f = \frac{1}{2}ma$ when there is rolling, and so the maximum acceleration possible with rolling is given by:

$a_{max} = \frac{2f_{max}}{m} = \frac{2\mu mg}{m} = 2\mu g$

(E) Wheel rolling at constant speed

If a wheel is rolling at constant speed, then there is no friction at the contact point: As there is no angular acceleration, there is no force for the friction to oppose. Also, ignoring resistance forces, any friction would be applying a translational force to the wheel (and this cannot be the case, as it is moving with constant speed).