

**Reference (Pure) (6 pages; 14/7/20)****(A) Definitions**

Whole numbers: integers

Natural numbers: no universal agreement;  
usually  $1, 2, 3, \dots$  (ie  $\mathbb{Z}^+$ ) but sometimes  $0, 1, 2, \dots$

**(B) Numbers**

$$\sqrt{2} = 1.4142135623730950488016887 \dots$$

$$e = 2.7182818284590452353602874 \dots$$

$$\text{Golden ratio: } \frac{1+\sqrt{5}}{2} = 1.618 \text{ (4sf)}$$

**(C) Series & expansions****(1) Summations**

$$\sum_{r=1}^n r = \frac{1}{2}n(n+1); \quad \sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$$

**(2) (i) For odd and even  $n$ :**

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \dots + xy^{n-2} + y^{n-1})$$

**(ii) For odd  $n$  only:**

$$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + \dots - xy^{n-2} + y^{n-1})$$

$$(3) (i) (a + b + c)^3 = (a^3 + b^3 + c^3)$$

$$+3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b)$$

+6abc

$$\begin{aligned} \text{(ii)} \quad (a + b + c)^4 &= (a^4 + b^4 + c^4) \\ &+ 4(a^3b + a^3c + b^3a + b^3c + c^3a + c^3b) \\ &+ 6(a^2b^2 + a^2c^2 + b^2c^2) + 12(a^2bc + b^2ac + c^2ab) \end{aligned}$$

$$\text{(iii)} \quad (a + b + c)^n = \sum_{(i+j+k=n)} \binom{n}{i,j,k} a^i b^j c^k,$$

$$\text{where } \binom{n}{i,j,k} = \frac{n!}{i!j!k!}$$

#### (4) Taylor expansions

$$\text{(i) Maclaurin: } f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots$$

$$\text{(ii) Taylor I: } f(x) = f(a) + (x - a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$$

$$\text{(iii) Taylor II: } f(x + a) = f(a) + xf'(a) + \frac{x^2}{2!} f''(a) + \dots$$

[x = 0 gives the Maclaurin expansion]

### (D) Trigonometry

#### (1) Powers of Sines and Cosines

[To deduce  $\sin^n \theta$  from  $\cos^n \theta$ :

(a) write  $\sin^n \theta = \cos^n(90 - \theta)$  etc, or

(b) employ the following pattern:

(i) the coefficients are the same as for  $\cos^n \theta$ , apart from signs

(ii) the 1st term within the brackets is

$\sin \theta$  ;  $-\cos 2\theta$  ;  $-\sin 3\theta$  ;  $\cos 4\theta$  ;  $\sin 5\theta$  etc

memory aid: integrate to get to the next one (without applying the Chain rule!)

(iii) the signs of the terms within the brackets alternate]

$$\cos^3 \theta = \frac{1}{4}(\cos 3\theta + 3\cos \theta)$$

$$\sin^3 \theta = \frac{1}{4}(-\sin 3\theta + 3\sin \theta)$$

$$\cos^4 \theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

$$\sin^4 \theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$$

$$\cos^5 \theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos \theta)$$

$$\sin^5 \theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin \theta)$$

$$\cos^6 \theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$$

$$\sin^6 \theta = \frac{1}{32}(-\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10)$$

$$\cos^7 \theta = \frac{1}{64}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos \theta)$$

$$\sin^7 \theta = \frac{1}{64}(-\sin 7\theta + 7\sin 5\theta - 21\sin 3\theta + 35\sin \theta)$$

(2)  $\cos(n\theta), \sin(n\theta)$

[ $\sin(2m\theta)$  can't be expressed in terms of powers of  $\sin\theta$ ]

$$\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$$

$$\sin 3\theta = -4\sin^3 \theta + 3\sin \theta$$

$$\cos 4\theta = 8\cos^4 \theta - 8\cos^2 \theta + 1$$

$$\cos 5\theta = 16\cos^5 \theta - 20\cos^3 \theta + 5\cos \theta$$

$$\sin 5\theta = 16\sin^5 \theta - 20\sin^3 \theta + 5\sin \theta$$

$$\cos 6\theta = 32\cos^6 \theta - 48\cos^4 \theta + 18\cos^2 \theta - 1$$

$$\cos 7\theta = 64\cos^7 \theta - 112\cos^5 \theta + 56\cos^3 \theta - 7\cos \theta$$

$$\sin 7\theta = -64\sin^7\theta + 112\sin^5\theta - 56\sin^3\theta + 7\sin\theta$$

## (E) Functions

(1) Reflection in the line  $x = a$ :  $f(x) \rightarrow f(2a - x)$

(2) For the cubic  $f(x) = ax^3 + bx^2 + cx + d$ :

(i) There is always one point of inflexion, at  $x = -\frac{b}{3a}$

(ii) Cubic curves have rotational symmetry (of order 2) about the PoI.

(iii) The ( $x$ -coordinate of the) PoI lies midway between any turning points.

(iv) The ( $x$ -coordinate of the) PoI is the average of the roots, when there are 3 real roots (and also when there are complex roots).

(v) There will be two turning points when  $b^2 > 3ac$

## (F) Geometry etc

(1) Tangents and normals to conics

(i) Parabola  $y^2 = 4ax$  at  $(at^2, 2at)$

$$\text{tangent: } y = \frac{1}{t}x + at$$

$$\text{normal: } y = -tx + 2at + at^3$$

(ii) Rectangular hyperbola  $xy = c^2$

$$\text{tangent: } y = -\frac{1}{t^2}x + \frac{2c}{t}$$

$$\text{normal: } y = t^2x + \frac{c}{t} - ct^3$$

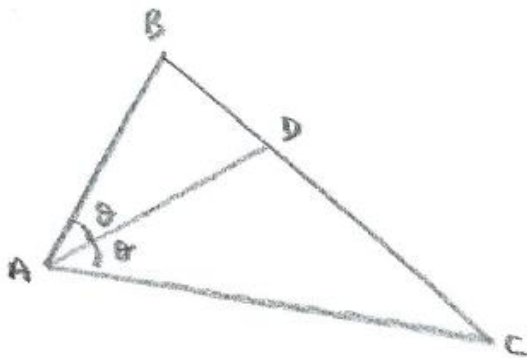
## (2) Areas &amp; Volumes

(i) Area of sector:  $\frac{1}{2}r^2\theta$ (consider limit of area of triangle  $\frac{1}{2}r^2\sin\theta$  as  $\theta \rightarrow 0$ )(ii) Volume of sphere:  $\frac{4}{3}\pi r^3$ (iii) Volume of pyramid or cone:  $\frac{1}{3} \times \text{base area} \times \text{height}$ 

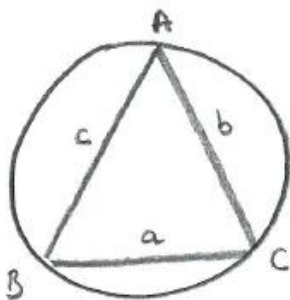
## (3) Angle Bisector Theorem

Referring to the diagram below, the Angle Bisector theorem says that

$$\frac{BD}{DC} = \frac{AB}{AC}$$



(4) ABC is a triangle circumscribed by a circle of radius R, as shown in the diagram below.



As an extension of the Sine rule,  $\frac{a}{\sin A} = 2R$

(5) Heron's formula for the area of a triangle with sides  $a, b$  &  $c$ :

$$\sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \frac{1}{2}(a+b+c)$$

### (G) Hyperbolic Functions

$$\cosh^2 x + \sinh^2 x = \cosh 2x; \quad \cosh^2 x - \sinh^2 x = 1$$

$$\operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1}); \quad \operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1}) \quad (x \geq 1)$$

$$\operatorname{artanh} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \quad (|x| < 1)$$

### (H) Calculus

(1) Derivatives

$$\frac{d}{dx}(a^x) = \ln a \cdot a^x$$

### (I) Numerical Methods

(1) Simpson's rule

$$\int_a^b y \, dx \approx$$

$$\frac{h}{3} \{(y_0 + y_n) + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})\}$$

where  $h = \frac{b-a}{n}$  ( $n$  even)