Reference (Pure) (7 pages; 4/1/21)

(A) Definitions

Whole number

There is no universal agreement about this. Mathematicians usually intend the term to mean "integer" (ie including zero and negative integers), but sometimes "positive integer" is intended (for example, if the context is Number Theory).

Natural number

Again, there is no universal agreement. Usually it means "positive integer", but sometimes zero is included.

(B) Numbers

$\sqrt{2} = 1.4142135623730950488016887 \ldots$

$e = 2.7182818284590452353602874 \ldots$

Golden ratio: $\frac{1+\sqrt{5}}{2} = 1.618$ (4sf)

(C) Series & expansions

(1) Summations

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$; $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

(2)(i) $x^2 - y^2 = (x-y)(x+y)$

(ii) $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

[Let $f(x) = x^3 - y^3$. Then $f(y) = 0$, and so $x - y$ is a factor of $x^3 - y^3$, by the Factor Theorem.]

(iii) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

or $(x + y)(x^{n-1} - x^{n-2}y + \cdots + xy^{n-2} - y^{n-1})$, if n is even

$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1})$ if n is odd

Summary

Factor:	$x - y$	$x + y$
$x^n - y^n$; odd n	Yes (A)	No **(B)**
$x^n - y^n$; even n	Yes (C)	Yes (D)
$x^n + y^n$; odd n	No (P)	Yes **(Q)**
$x^n + y^n$; even n	No (R)	No (S)

[The 'exceptions' are highlighted. As an aid to memory, the familiar factorisations $x^2 - y^2 = (x - y)(x + y)$ and

$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ are examples of (C), (D) & (A), suggesting that $x^3 - y^3 = (x + y)$ ... (ie (B)) is the one that isn't possible for $x^n - y^n$. This then prompts us to recall that

$x^3 + y^3 = (x + y)$ ... (ie (Q)) is the one that is possible for

$x^n + y^n$.]

(3) (i) $(a + b + c)^3 = (a^3 + b^3 + c^3)$

$+3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b)$

$+6abc$

(ii) $(a + b + c)^4 = (a^4 + b^4 + c^4)$

$+4(a^3b + a^3c + b^3a + b^3c + c^3a + c^3b)$

$+6(a^2b^2+a^2c^2+b^2c^2)+12(a^2bc+b^2ac+c^2ab)$

(iii) $(a+b+c)^n=\sum_{\substack{i,j,k \\ (i+j+k=n)}}\binom{n}{i,j,k}a^ib^jc^k$,

where $\binom{n}{i,j,k}=\frac{n!}{i!j!k!}$

(4) Taylor expansions

(i) Maclaurin: $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\cdots$

(ii) Taylor I: $f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots$

(iii) Taylor II: $f(x+a)=f(a)+xf'(a)+\frac{x^2}{2!}f''(a)+\cdots$

[$x=0$ gives the Maclaurin expansion]

(D) Trigonometry

(1) Powers of Sines and Cosines

[To derive $sin^n\theta$ from $cos^n\theta$, write $sin^n\theta=cos^n(90-\theta)$ etc]

$cos^3\theta=\frac{1}{4}(cos3\theta+3cos\theta)$

$sin^3\theta=\frac{1}{4}(-sin3\theta+3sin\theta)$

$cos^4\theta=\frac{1}{8}(cos4\theta+4cos2\theta+3)$

$sin^4\theta=\frac{1}{8}(cos4\theta-4cos2\theta+3)$

$cos^5\theta=\frac{1}{16}(cos5\theta+5cos3\theta+10cos\theta)$

$sin^5\theta=\frac{1}{16}(sin5\theta-5sin3\theta+10sin\theta)$

$cos^6\theta=\frac{1}{32}(cos6\theta+6cos4\theta+15cos2\theta+10)$

$sin^6\theta = \frac{1}{32}(-cos6\theta + 6cos4\theta - 15cos2\theta + 10)$

$cos^7\theta = \frac{1}{64}(cos7\theta + 7cos5\theta + 21cos3\theta + 35cos\theta)$

$sin^7\theta = \frac{1}{64}(-sin7\theta + 7sin5\theta - 21sin3\theta + 35sin\theta)$

(2) $cos(n\theta), sin(n\theta)$

[sin($2m\theta$) can't be expressed in terms of powers of $sin\theta$]

$cos3\theta = 4cos^3\theta - 3cos\theta$

$sin3\theta = -4sin^3\theta + 3sin\theta$

$cos4\theta = 8cos^4\theta - 8cos^2\theta + 1$

$cos5\theta = 16cos^5\theta - 20cos^3\theta + 5cos\theta$

$sin5\theta = 16sin^5\theta - 20sin^3\theta + 5sin\theta$

$cos6\theta = 32cos^6\theta - 48cos^4\theta + 18cos^2\theta - 1$

$cos7\theta = 64cos^7\theta - 112cos^5\theta + 56cos^3\theta - 7\cos\theta$

$sin7\theta = -64sin^7\theta + 112sin^5\theta - 56sin^3\theta + 7sin\theta$

(E) Functions

(1) Reflection in the line $x = a$: $f(x) \rightarrow f(2a - x)$

(2) For the cubic $f(x) = ax^3 + bx^2 + cx + d$:

(i) There is always one point of inflexion, at $x = -\frac{b}{3a}$

(ii) Cubic curves have rotational symmetry (of order 2) about the PoI.

(iii) The (x-coordinate of the) PoI lies midway between any turning points.

(iv) The (x-coordinate of the) PoI is the average of the roots, when there are 3 real roots (and also when there are complex roots).

(v) There will be two turning points when $b^2 > 3ac$

(F) Geometry & Solids

(1) Tangents and normals to conics

(i) Parabola $y^2 = 4ax$ at $(at^2, 2at)$

tangent: $y = \frac{1}{t}x + at$

normal: $y = -tx + 2at + at^3$

(ii) Rectangular hyperbola $xy = c^2$

tangent: $y = -\frac{1}{t^2}x + \frac{2c}{t}$

normal: $y = t^2x + \frac{c}{t} - ct^3$

(2) Areas & Volumes

(i) Area of sector: $\frac{1}{2}r^2\theta$

(consider limit of area of triangle $\frac{1}{2}r^2 sin\theta$ as $\theta \to 0$)

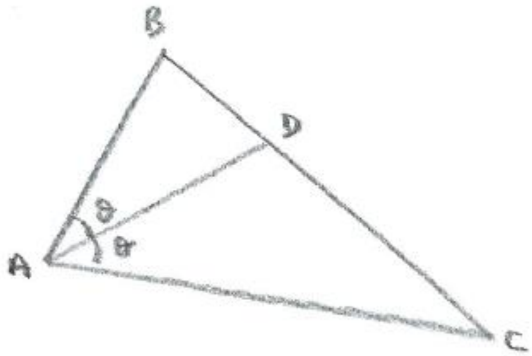
(ii) Volume of sphere: $\frac{4}{3}\pi r^3$

(iii) Volume of pyramid or cone: $\frac{1}{3} \times$ base area $\times$ height

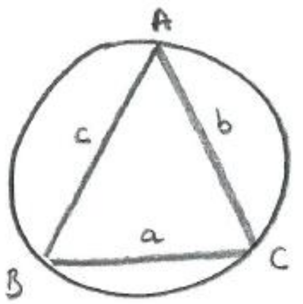
(3) Angle Bisector Theorem

Referring to the diagram below, the Angle Bisector theorem says that

$\frac{BD}{DC} = \frac{AB}{AC}$



(4) ABC is a triangle circumscribed by a circle of radius R, as shown in the diagram below.



As an extension of the Sine rule, $\frac{a}{sinA} = 2R$

(5) Heron's formula for the area of a triangle with sides a, b & c:

$\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$

(G) Hyperbolic Functions

$cosh^2x + sinh^2x = cosh2x$; $cosh^2x - sinh^2x = 1$

$arsinhx = \ln\left(x + \sqrt{x^2+1}\right)$; $arcoshx = \ln\left(x + \sqrt{x^2-1}\right)$ $(x \geq 1)$

$artanhx = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ $(|x| < 1)$

(H) Calculus

(1) Derivatives

$\frac{d}{dx}(a^x) = lna.a^x$

(I) Numerical Methods

(1) Simpson's rule

$\int_a^b y\,dx \approx$

$\frac{h}{3}\{(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\}$

where $h = \frac{b-a}{n}$ $(n$ even)