Recurrence relations (16 pages; 5/9/18)

(1) Terminology

The arithmetic sequence $2, 6, 10, 14, 18, \ldots$ can be represented by the **deductive** (or closed) definition $u_n = 4n - 2$, or by the **recurrence relation** (also known as an inductive or recursive definition) $u_n = u_{n-1} + 4,\ u_1 = 2$.

Similarly, the deductive definition for the geometric sequence

$2, 6, 18, 54, 162, \ldots$ is $u_n = 2(3)^{n-1}$, whilst the equivalent recurrence relation is $u_n = 3u_{n-1},\ u_1 = 2$.

Another recurrence relation is $u_n = 3u_{n-1} + 4,\ u_1 = 2$ (which can be thought of as a combination of the arithmetic and geometric sequences).

A **homogeneous** recurrence relation is one such as $u_n = 3u_{n-1}$ or

$u_n = 3u_{n-1} + 2u_{n-2}$ ('homogeneous' refers to the fact that the terms are all of the same form; ie au_{n-r}), whilst a

non-homogeneous recurrence relation is one such as

$u_n = 3u_{n-1} + 4$, or $u_n = u_{n-1} - 2u_{n-2} + 3n$, where there is a term not of the form au_{n-r}.

A **linear** recurrence relation is one which doesn't involve any quadratic or higher powers of u_{n-r} (for $r = 0, 1, 2, \ldots$).

A **first order** recurrence relation is one which doesn't involve u_{n-2}, u_{n-3} etc. The recurrence relation $u_n = u_{n-1} + u_{n-3}$ would be described as third order, for example.

In order to fully specify a recurrence relation, a first order equation would require an **initial condition** such as $u_1 = 2$ (as we have seen). For an nth order equation there would need to be n such conditions (eg for a 2nd order equation, $u_1 = 2$ and $u_2 = 3$).

[Note that 'equation' is often used in place of 'recurrence relation'.]

Note, by the way, that an equation may be defined for $n \geq 0$, so that an initial condition may take the form $u_0 = 2$, for example.

Sequences may be **convergent** (having a **limit** as $n \to \infty$) or divergent. In the case of convergent sequences, the limit L may be found by replacing each u_{n-r} term (for $r = 0, 1, 2, \ldots$) by L (assuming that the resulting equation can be solved).

Note that the term 'divergent' applies to periodic or oscillating sequences (ie not just sequences where the terms get progressively larger in magnitude).

Also note that the method for finding L should only be applied once it is clear (from the terms of the sequence) that there is convergence, as illustrated by the recurrence relation

$u_n = 2u_{n-1} - 1, u_1 = 2 \ \ (2, 3, 5, 9, 17, \ldots)$

In this case, the equation $L = 2L - 1$ has the solution $L = 1$, which is only the limit when $u_1 = 1$, and all the terms are 1.

(2) Linear, homogeneous equations

[The approach adopted in this, and subsequent sections, is very similar to that employed when solving differential equations.]

Note that the solution to $u_n = 3u_{n-1},\ u_1 = 2$ was $u_n = 2(3)^{n-1}$.

This can be written in the form $u_n = kx^n$ (with $x = 3$ & $k = \frac{2}{3}$).

This is the basis for considering $u_n = x^n$, as a possible solution for the 2nd order **homogeneous** equation

$u_n = 5u_{n-1} - 6u_{n-2}$, for example.

Substituting into the equation gives

$x^n = 5x^{n-1} - 6x^{n-2}$

Excluding the trivial solution where $x = 0$, this gives

$x^2 - 5x + 6 = 0$ (the **auxiliary** equation)

so that $(x-2)(x-3) = 0,$ and $x = 2$ or 3.

Thus two possible solutions are $u_n = 2^n$ and 3^n.

In general, it can be seen that, if $u_n = {x_1}^n$ and ${x_2}^n$ are solutions, then $A{x_1}^n + B{x_2}^n$ will also be a solution.

It can also be shown that, for a 2nd order equation, this is the most general solution; ie all solutions are of this form.

Similarly, for an nth order equation, the most general solution is a linear combination (with n constants) of solutions of an nth order auxiliary equation.

As seen above, the auxiliary equation can be written down straightaway from the recurrence relation.

Thus $u_n = au_{n-1} + bu_{n-2}$ produces $x^2 = ax + b$

In the case of a first order equation, the auxiliary equation is very simple. Applying it to $u_n = 3u_{n-1}$ (considered earlier), we obtain just $x = 3$.

Once a general solution has been obtained, the initial conditions are then applied, to find a specific solution to the recurrence relation. Thus, if the initial conditions of the equation

$u_n = 5u_{n-1} - 6u_{n-2}$ are $u_1 = 2$ and $u_2 = 3$, the specific solution is found from substitution into the general solution $A2^n + B3^n$,

so that $A(2) + B(3) = 2$ and $A(2^2) + B(3^2) = 3$;

ie $2A + 3B = 2$ and $4A + 9B = 3$,

giving $A = \frac{3}{2}$ and $B = -\frac{1}{3}$,

so that $u_n = \left(\frac{3}{2}\right)2^n - (\frac{1}{3})3^n$ is the specific solution that satisfies the initial conditions.

Note that the expression 'specific solution' isn't universal, and sometimes 'particular solution' is used instead. This is unfortunate, because 'particular solution' has another meaning (as we shall see shortly). Usually the intended meaning is clear from the context though.

The above approach is very similar to that employed when solving differential equations.

Example: $u_n = u_{n-1} + u_{n-2}$; $u_0 = 0, u_1 = 1$

This is the Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

[The u_0 term is usually omitted when quoting the sequence. However it helps in the working below.]

The auxiliary equation is $x^2 - x - 1$, which has roots

$x = \frac{1\pm\sqrt{5}}{2}$ (ϕ and $\overline{\phi}$, say)

The general solution is therefore $u_n = A\phi^n + B\overline{\phi}^n$

Applying the initial conditions,

$A + B = 0$ and $A\phi + B\overline{\phi} = 1$

[Note that having u_0 & u_1 as the initial values, rather than u_1 & u_2, simplifies the working.]

Thus $\frac{1}{2}A\left(1 + \sqrt{5}\right) + \frac{1}{2}(-A)\left(1 - \sqrt{5}\right) = 1$, so that $A(2\sqrt{5}) = 2$, and $A = \frac{1}{\sqrt{5}}$, $B = -\frac{1}{\sqrt{5}}$

So the specific solution is $u_n = \frac{1}{\sqrt{5}}(\phi^n - \overline{\phi}^n)$

Notes

(i) ϕ is the 'golden ratio', which can be defined as follows:

A 'golden rectangle', having sides in the ratio of ϕ, is such that it can be divided into a square and another golden rectangle. Without loss of generality, the sides can then be taken as being of lengths ϕ and $\phi + 1$, such that $\frac{\phi+1}{\phi} = \phi$, giving $\phi^2 - \phi - 1 = 0$.

(ii) $\phi\overline{\phi} = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right) = \frac{1-5}{4} = -1$, so that $\overline{\phi}$ can also be written as $-\frac{1}{\phi}$

(iii) It can be shown that the ratio of successive terms of the Fibonacci sequence converges to ϕ:

(a) $u_n = u_{n-1} + u_{n-2}$

$\Rightarrow \frac{u_n}{u_{n-1}} = 1 + \frac{u_{n-2}}{u_{n-1}}$

If the ratio tends to λ, $\lambda = 1 + \frac{1}{\lambda} \Rightarrow \lambda^2 = \lambda + 1 \Rightarrow \lambda^2 - \lambda - 1 = 0$,

so that $\lambda = \phi$.

(b) $\frac{u_{n+1}}{u_n} = \frac{\frac{1}{\sqrt{5}}(\phi^{n+1} - \overline{\phi}^{n+1})}{\frac{1}{\sqrt{5}}(\phi^n - \overline{\phi}^n)} = \frac{\phi - \phi\left(\frac{-1/\phi}{\phi}\right)^{n+1}}{1 - \left(\frac{-1/\phi}{\phi}\right)^n} = \frac{\phi - \phi\left(\frac{-1}{\phi^2}\right)^{n+1}}{1 - \left(\frac{-1}{\phi^2}\right)^n} \to \frac{\phi - 1}{1 - 0} = \phi$

(3) Linear, homogeneous equations (cont'd): complex roots of auxiliary equation

Example: $u_n = 4u_{n-1} - 13u_{n-2}$; $u_1 = 1, u_2 = 2$

The auxiliary equation is $x^2 - 4x + 13$,

which has the complex roots $x = \frac{4 \pm \sqrt{16-52}}{2} = 2 \pm 3i$

and gives the general solution $u_n = A(2+3i)^n + B(2-3i)^n$ (*)

It will be seen in a moment that, despite appearances, this can produce a real solution.

$2 + 3i$ can be written as $\sqrt{13}(cos\theta + isin\theta)$, where $tan\theta = \frac{3}{2}$

Then, by de Moivre's theorem,

$u_n = A(13)^{\frac{n}{2}}\big(\cos(n\theta) + isin(n\theta)\big) + B(13)^{\frac{n}{2}}(\cos(n\theta) - isin(n\theta))$

Write $C = A + B$ & $D = i(A - B)$,

so that $u_n = C(13)^{\frac{n}{2}}\cos(n\theta) + D(13)^{\frac{n}{2}}\sin(n\theta)$

This show that a real solution will exist, and can be chosen to satisfy the initial conditions (because of the two arbitrary constants).

To find the values of C & D, it is more convenient to use the cartesian form of the complex numbers:

Applying the initial conditions to (*),

$1 = A(2+3i) + B(2-3i)$ (1)

and $2 = A(2+3i)^2 + B(2-3i)^2$

$= A(4 + 12i - 9) + B(4 - 12i - 9)$

$= A(-5 + 12i) + B(-5 - 12i)$ (2)

Then write $C = A + B$ & $D = i(A - B)$,

so that (1) & (2) give

$1 = 2C + 3D$ (3) and $2 = -5C + 12D$ (4)

$4 \times (3) - (4)$ then gives $2 = 13C$,

so that $C = \frac{2}{13}$ and $D = \frac{1}{3}\left(1 - \frac{4}{13}\right) = \frac{3}{13}$

[As $iD = B - A$, and hence $C + iD = 2B$ & $C - iD = 2A$, this corresponds to $A = \frac{1}{26}(2 - 3i)$ & $B = \frac{1}{26}(2 + 3i)$]

So the specific solution is

$u_n = \frac{2}{13}(13)^{\frac{n}{2}}\cos(n\theta) + \frac{3}{13}(13)^{\frac{n}{2}}\sin(n\theta)$

(4) Linear, homogeneous equations (cont'd): repeated roots of auxiliary equation

Example: $u_n = 4u_{n-1} - 4u_{n-2}$ are $u_1 = 1$ and $u_2 = 5$

Here the auxiliary equation is $x^2 - 4x + 4 = 0$, or $(x - 2)^2 = 0$,

and this has the repeated root $x = 2$.

As before, $u_n = A(2)^n$ is a solution. However, it isn't sufficiently general for a 2nd order relation, as it only involves one arbitrary constant.

However, we can show that $u_n = Bn(2)^n$ will be a solution.

Consider the general 2nd order case:

$u_n + bu_{n-1} + cu_{n-2} = 0$,

where λ is the repeated root of the auxiliary equation.

Then consider $u_n = Bn\lambda^n$:

$u_n + bu_{n-1} + cu_{n-2} = Bn\lambda^n + bB(n-1)\lambda^{n-1} + cB(n-2)\lambda^{n-2}$

$= Bn(\lambda^n + b\lambda^{n-1} + c\lambda^{n-2}) - B\lambda^{n-2}[b\lambda + 2c]$

The 1st expression is zero, as λ is a root of the auxiliary equation.

Also, the sum and product of the roots of the auxiliary equation are $-\frac{b}{1}$ and $\frac{c}{1}$, so that $2\lambda = -b$ and $\lambda^2 = c$,

and hence $b\lambda + 2c = (-2\lambda)\lambda + 2\lambda^2 = 0$

Thus $Bn\lambda^n$ is a solution of the recurrence relation.

It can be shown that the most general solution is of the form

$(A + Bn)\lambda^n$.

For our example, we have $u_n = (A + Bn)(2)^n$, with $u_1 = 1$ and $u_2 = 5$, so that:

$1 = (A + B)(2)$ and $5 = (A + 2B)(4)$,

or $2 = 4A + 4B$ and $5 = 4A + 8B$,

so that $3 = 4B$, giving $B = \frac{3}{4}$ and $A = -\frac{1}{4}$

Thus the solution of the recurrence relation is

$u_n = \frac{1}{4}(3n - 1)(2)^n$

(5) Linear, non-homogeneous equations

This covers cases of the form $u_n = au_{n-1} + f(n)$

or $u_n = au_{n-1} + bu_{n-2} + f(n)$ etc

Example: $u_n = 5u_{n-1} - 6u_{n-2} + n^2 + 2n$; $u_1 = 1, u_2 = 4$

The homogeneous equation $u_n = 5u_{n-1} - 6u_{n-2}$ was dealt with in (2), and the general solution was found to be

$u_n = A(2^n) + B(3^n)$. This is referred to as the **complementary function** (CF), and will turn out to be part of the solution of the non-homogeneous equation (as will become clear shortly).

[The initial conditions are different, but this doesn't need to be the case.]

Suppose that we can find a **particular solution** of

$u_n = 5u_{n-1} - 6u_{n-2} + n^2 + 2n$ (which doesn't need to satisfy our initial conditions). Let this particular solution be $u_n = g(n)$.

[Note: As already mentioned, the expression 'particular solution' is sometimes used (ill-advisedly) to mean a specific solution (usually of a homogeneous equation); ie one that satisfies the initial conditions. The problem doesn't arise when solving differential equations (in a very similar way): in that case, the expression 'particular integral' is used instead of 'particular solution'.]

We can rewrite the equation as

$u_n - 5u_{n-1} + 6u_{n-2} = n^2 + 2n$ (1)

Then, when $u_n = A(2^n) + B(3^n)$, the left-hand side is 0,

and, when $u_n = g(n)$, the left-hand side is $n^2 + 2n$.

Hence, when $u_n = A(2^n) + B(3^n) + g(n)$, the left-hand side is also $n^2 + 2n$.

Because this function has two arbitrary constants, it can be shown that it is the most general solution of the (2nd order) equation.

We can then choose values for A and B, such that the initial conditions are satisfied;

ie such that $A(2^1) + B(3^1) + g(1) = 1$

and $A(2^2) + B(3^2) + g(2) = 4$

The particular function can be found by past experience, and the following table shows the trial function that can be substituted into the equation, for commonly occurring forms of $f(n)$.

$f(n)$	trial function
a	p
$an(+b)$	$pn+q$
$an^2(+bn+c)$	pn^2+qn+r
ak^n	pk^n (or pnk^n if the CF includes Ak^n)
$(an^2[+bn+c])k^n$	$(pn^2+qn+r)k^n$

In this example, we take $u(n) = pn^2 + qn + r$ as the trial function, giving:

$$pn^2 + qn + r = 5\{p(n-1)^2 + q(n-1) + r\}$$
$$-6\{p(n-2)^2 + q(n-2) + r\} + n^2 + 2n$$

Equating coefficients of n^2 gives:

$p = 5p - 6p + 1 \Rightarrow p = \frac{1}{2}$

Equating coefficients of n gives:

$q = -10p + 5q + 24p - 6q + 2$

$\Rightarrow 2q = 7 + 2 \Rightarrow q = \frac{9}{2}$

And equating the constant terms gives:

$r = 5p - 5q + 5r - 24p + 12q - 6r$

$\Rightarrow 2r = -\frac{19}{2} + \frac{63}{2} \Rightarrow r = 11$

So the particular solution $g(n) = \frac{1}{2}n^2 + \frac{9}{2}n + 11$,

and the general solution is

$u_n = A(2^n) + B(3^n) + \frac{1}{2}n^2 + \frac{9}{2}n + 11$

Applying the initial conditions $u_1 = 1, u_2 = 4$ then gives

$A(2^1) + B(3^1) + \frac{1}{2}1^2 + \frac{9}{2}(1) + 11 = 1$

and $A(2^2) + B(3^2) + \frac{1}{2}2^2 + \frac{9}{2}(2) + 11 = 4$.

so that $2A + 3B = -15$ and $4A + 9B = -18$,

which produces $A = -\frac{27}{2}$ & $B = 4$

Thus, the specific solution is

$u_n = 4(3^n) - \frac{27}{2}(2^n) + \frac{1}{2}n^2 + \frac{9}{2}n + 11$

Alternative approach

Instead of finding the particular solution separately, an alternative approach that is sometimes advocated is to combine this with the process of applying the initial conditions, using the recurrence relation to generate further conditions that have to be satisfied. For the example above,

$u_n = A(2^n) + B(3^n) + pn^2 + qn + r$

As well as applying the initial conditions $u_1 = 1, u_2 = 4$, we know from the recurrence relation $u_n - 5u_{n-1} + 6u_{n-2} = n^2 + 2n$ that

$u_3 = 5(2) - 6(1) + 9 + 6 = 19$, and similarly for u_4 and u_5.

However, for this example, solving the 5 simultaneous equations will probably be a longer process. Also, strictly speaking, we won't have shown that $pn^2 + qn + r$ satisfies the recurrence relation for all n.

(6) Linear, non-homogeneous equations - special cases

The complementary function will be of the form $Ak^n + Bl^n$, and it may be the case that the right-hand side of the non-homogeneous equation is ak^n. If this is the case, then the particular solution cannot be pk^n, as this would generate zero for the right-hand side of (1), instead of $f(n)$ (as pk^n is the CF with $A = p$, and $B = 0$).

In this situation it can be shown that an appropriate trial function is pnk^n.

Exercise: Show that this is the case for a 2nd order equation.

Solution

Consider the recurrence equation

$u_n - (k + l)u_{n-1} + klu_{n-2} = ak^n$

where the auxiliary equation has roots k & l

We can investigate the trial function $u_n = pnk^n$.

Substituting into the equation gives

$pnk^n - (k + l)(p[n - 1]k^{n-1}) + kl(p[n - 2]k^{n-2}) = ak^n$

$\Rightarrow (pn - a)k^2 - (k + l)p(n - 1)k + klp(n - 2) = 0$

$\Rightarrow k\{pn - a - p(n - 1)\} - lp(n - 1) + lp(n - 2) = 0$

$\Rightarrow k(p - a) - lp = 0$

$\Rightarrow p(k - l) = ka$

$\Rightarrow p = \frac{ka}{k-l}$ (provided $k \neq l$)

Note: Because the choice of trial function can be influenced by the complementary function, it is best to find the complementary function before the particular solution.

Exercise: Find a suitable trial function for a 2nd order equation when $f(n) = ak^n$, and the auxiliary equation has repeated roots of k.

Solution

Consider the recurrence equation

$u_n - 2ku_{n-1} + k^2u_{n-2} = ak^n$

We can investigate the trial function $u_n = pn^2k^n$.

Substituting into the equation gives

$pn^2k^n - 2kp(n-1)^2k^{n-1} + k^2p(n-2)^2k^{n-2} = ak^n$

$\Rightarrow pn^2 - 2p(n-1)^2 + p(n-2)^2 - a = 0$

$\Rightarrow n(4p - 4p) + 4p - a = 0$

$\Rightarrow p = \frac{a}{4}$

(7) 1st order equations

These take the general form $u_n = au_{n-1} + b$

Note that when $a = 1$ & $b \neq 0$, u_n is an arithmetic sequence.

And when $a \neq 1$ & $b = 0$, it is a geometric sequence.

More generally, the usual method can be applied, to give an auxiliary equation of $x - a = 0$, so that the CF is $u_n = Aa^n$.

The trial function for the particular solution will be $u_n = c$,

giving $c = ac + b$, so that $c = \frac{b}{1-a}$,

and the general solution is $u_n = Aa^n + \frac{b}{1-a}$

Then, applying the initial condition u_0:

$u_0 = A + \frac{b}{1-a}$, so that the specific solution is

$u_n = \left(u_0 - \frac{b}{1-a}\right) a^n + \frac{b}{1-a}$ (*)

Thus, u_n is a linear function of a^n.

Alternative method

As an alternative to deriving (*) as above, we could use the fact that u_n is always a linear function of a^n to write

$u_n = \lambda a^n + \mu$, and then $u_0 = \lambda + \mu$ (1)

Also, $u_1 = au_0 + b$, so that $au_0 + b = \lambda a + \mu$ (2)

Then (2) − (1) gives $u_0(a-1) + b = \lambda(a-1)$,

so that $\lambda = u_0 + \frac{b}{a-1}$ or $u_0 - \frac{b}{1-a}$

and then $\mu = u_0 - \lambda = \frac{b}{1-a}$, as before.

(8) Calculators

Some calculators have a 'table' function to determine the series from the formula, and/or a recurrence function to determine the series from the recurrence relation.

(9) Generating Functions: homogeneous equations

The generating function for u_n is defined to be

$u(x) = u_0 + u_1 x + u_2 x^2 + \cdots$

Suppose that the recurrence equation is $u_n + au_{n-1} + bu_{n-2} = 0$

Then $u(x) + axu(x) + bx^2u(x)$

$= u_0 + u_1x + u_2x^2 + \cdots$

$+ au_0x + au_1x^2 + au_2x^3 + \cdots$

$+ bu_0x^2 + bu_1x^3 + bu_2x^4 + \cdots$

$= u_0 + x(u_1 + \ au_0) + x^2(u_2 + au_1 + \ bu_0)$

$+x^3(u_3 + au_2 + \ bu_1) + \cdots$

$= u_0 + x(u_1 + \ au_0)$, as all subsequent terms vanish

And hence $u(x) = \frac{u_0 + x(u_1 + au_0)}{1 + ax + bx^2}$ (3)

Example: Consider the recurrence equation from (2):

$u_n - 5u_{n-1} + 6u_{n-2} = 0$, with $u_1 = 2$ and $u_2 = 3$,

[for which the solution was found to be $u_n = \left(\frac{3}{2}\right)2^n - (\frac{1}{3})3^n$]

First of all, we can find u_0:

$u_2 - 5u_1 + 6u_0 = 0$, so that $u_0 = \frac{1}{6}(10 - 3) = \frac{7}{6}$

Then, from (3), we have $u(x) = \frac{\frac{7}{6} + x\left(2 - 5[\frac{7}{6}]\right)}{1 - 5x + 6x^2}$

$= \frac{7 - 23x}{6(3x-1)(2x-1)} = \frac{A}{6(3x-1)} + \frac{B}{6(2x-1)}$,

where $7 - 23x = A(2x - 1) + B(3x - 1)$

$x = \frac{1}{2} \Rightarrow -\frac{9}{2} = \frac{1}{2}B \Rightarrow B = -9$

and $x = \frac{1}{3} \Rightarrow -\frac{2}{3} = -\frac{1}{3}A \Rightarrow A = 2$

So $u(x) = \frac{1}{3(3x-1)} - \frac{3}{2(2x-1)}$

$= -\frac{1}{3}(1 - 3x)^{-1} + \frac{3}{2}(1 - 2x)^{-1}$

$= -\frac{1}{3}(1 + 3x + (3x)^2 + \cdots) + \frac{3}{2}(1 + 2x + (2x)^2 + \cdots)$

giving $u_n = -\frac{1}{3}(3)^n + \frac{3}{2}(2)^n$, as expected

(10) Generating Functions: non-homogeneous equations

The approach is the same, but not all the terms vanish on the right-hand side. Depending on the form of $f(n)$, it may or may not be possible to write the remaining terms as an expression in x.

Suppose that the recurrence equation is

$u_n + au_{n-1} + bu_{n-2} = f(n)$

Then, as before, $u(x) + axu(x) + bx^2u(x)$

$= u_0 + u_1x + u_2x^2 + \cdots$

$+ au_0x + au_1x^2 + au_2x^3 + \cdots$

$+ bu_0x^2 + bu_1x^3 + bu_2x^4 + \cdots$

$= u_0 + x(u_1 + au_0) + x^2(u_2 + au_1 + bu_0)$

$+x^3(u_3 + au_2 + bu_1) + \cdots$

$= u_0 + x(u_1 + au_0) + x^2f(2) + x^3f(3) + \cdots$

So, for the simple case where $f(n)$ is a constant, k say:

$u(x) = \frac{u_0 + x(u_1 + au_0) + kx^2(1-x)^{-1}}{1 + ax + bx^2}$, and this can be dealt with in the same way as before.