Projectiles (11 pages; 10/3/21)

(1) Introduction

A typical example is the trajectory of a golf ball.

The ball is modelled as a particle (so that any spinning is ignored).

In this example, the ball will be starting and finishing at ground level, but in some questions the starting point may be above ground level.

The horizontal and vertical components of the motion are independent, and the suvat equations can be applied to each of them.

Horizontally there is no acceleration, as air resistance is always ignored (as a simplifying assumption).

Vertically, if upwards is taken as the positive direction, the acceleration is $-g$.

Another simplifying assumption is that gravity does not vary with height (though this is quite minor, compared to the other two assumptions).

If we are given the initial velocity of the particle, the first step will be to determine its horizontal and vertical components.

Standard questions involve the following:

(a) the maximum height reached by the particle

(b) the time taken to reach that maximum height

(c) the 'time in flight': how long it takes before the particle hits the ground

(d) the 'range' of the particle: how far it travels horizontally before it hits the ground

(e) the speed of the particle on hitting the ground

(2) Example

Initial speed: $10ms^{-1}$

Angle of projection: 30° to horizontal

Starting at ground level

(i) To find the maximum height reached:

initial components of velocity:

horizontally: $u_x = 10cos30° = 10.\frac{\sqrt{3}}{2} = 5\sqrt{3}$

vertically $u_y = 10sin30° = 10.\frac{1}{2} = 5$

max. height, s_y occurs when $v_y = 0$

$v_y{}^2 = u_y{}^2 + 2(-g)s_y \Rightarrow 0 = 25 - 2gs_y$

$\Rightarrow s_y = \frac{25}{2g}\ m$

(ii) To find the time in flight:

The time to reach the maximum height can be found from

$v_y = u_y + (-g)t$

Thus $0 = 5 - gt$,

so that $t = \frac{5}{g}$

Then, by the symmetrical nature of the path of the object,

time in flight $= 2\left(\frac{5}{g}\right) = \frac{10}{g}s$

(iii) To find the range:

$5\sqrt{3}\left(\frac{10}{g}\right) = \frac{50\sqrt{3}}{g}\ m$

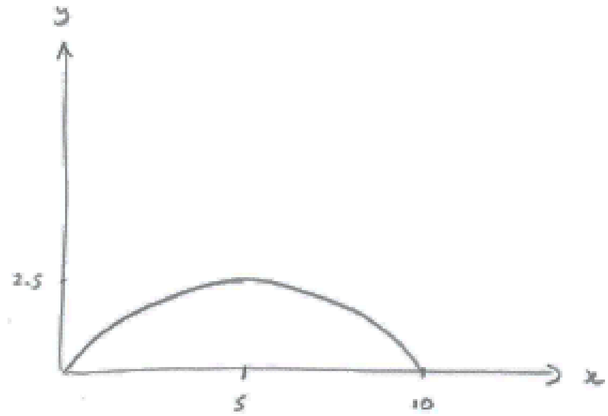
(3) Cartesian equation

(i) A cartesian equation for the trajectory (ie with y as a function of x) can be derived from $y = usin\theta.t - \frac{1}{2}gt^2$ and $x = ucos\theta.t$, by eliminating t, to give $y = xtan\theta - \frac{gx^2sec^2\theta}{2u^2}$

Note incidentally that y is a quadratic function of x, as well as being a quadratic function of t.

(ii) **Example**: $y = x - 0.1x^2$

$\Rightarrow\ y = x(1 - 0.1x)$



Comparing with $y = xtan\theta - \frac{gx^2sec^2\theta}{2u^2}$, we see that

$tan\theta = 1 \Rightarrow sec^2\theta = tan^2\theta + 1 = 2$

$\Rightarrow\ \frac{9.8}{u^2} = 0.1\ \ \Rightarrow u = 9.90\ ms^{-1}$

(iii) The cartesian equation leads to an expression for the range:

$xtan\theta \; - \frac{gx^2sec^2\theta}{2u^2} = 0 \Rightarrow$

either $x = 0$ or $tan\theta - \frac{gxsec^2\theta}{2u^2} = 0$

$\Rightarrow x = tan\theta.\frac{2u^2}{gsec^2\theta} = \frac{2sin\theta cos\theta.u^2}{g} = \frac{sin2\theta.u^2}{g}$

To maximise the range (for a given u) , we therefore want

$\theta = 45°$ (giving $sin2\theta = 1$).

Also, note that the range for angle θ is the same as that for angle $\frac{\pi}{2} - \theta$ (as $sin2\left(\frac{\pi}{2} - \theta\right) = \sin(\pi - 2\theta) = \sin\,(2\theta)$). Thus there are always two possible trajectories to achieve a particular range - except when $\theta = 45°$.

(4) Alternative methods for the example in (2).

(i) From $s_y = u_yt + \frac{1}{2}(-g)t^2$, having already determined t:

maximum height $= 5.\frac{5}{g} - \frac{1}{2}g\left(\frac{5}{g}\right)^2 = \frac{25}{2g}$

(ii) $s_y = \frac{1}{2}\left(u_y + v_y\right)t \Rightarrow maximum\ height = \frac{1}{2}(5 + 0)\left(\frac{5}{g}\right) = \frac{25}{2g}$

(iii) $s_y = v_yt - \frac{1}{2}(-g)t^2$

$\Rightarrow maximum\ height = 0 + \frac{1}{2}g\left(\frac{5}{g}\right)^2 = \frac{25}{2g}$

(iv) Object hits the ground when $s_y = 0$,

so that, from $s_y = u_y t + \frac{1}{2}(-g)t^2$,

$0 = 5t - \frac{1}{2}gt^2$

$\Rightarrow 0 = t(10 - gt)$

$\Rightarrow t = 0$ (at the start) or $t = \frac{10}{g}$

(v) By symmetry, $v_y = -u_y$,

so that $v_y = u_y + (-g)t \Rightarrow -2u_y = (-g)t$ and $t = \frac{10}{g}$

(vi) As $cos\theta = \frac{\sqrt{3}}{2}$, $sec^2\theta = \frac{1}{\left(\frac{3}{4}\right)} = \frac{4}{3}$

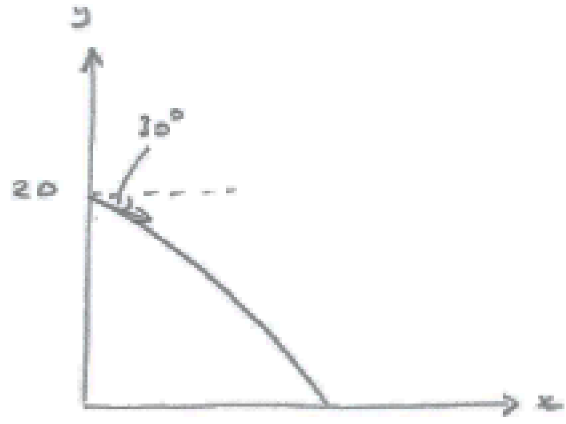
The cartesian equation is $y = \frac{x}{\sqrt{3}} - \frac{gx^2\left(\frac{4}{3}\right)}{2(100)} = \frac{x}{\sqrt{3}} - \frac{gx^2}{150}$

To find the range: $0 = \frac{x}{\sqrt{3}} - \frac{gx^2}{150}$

$\Rightarrow x = 0$ or $x = \frac{150}{g\sqrt{3}} = \frac{50\sqrt{3}}{g}$

(5) Example

$u = 10ms^{-1}$ downwards at 30°, from a height of $20m$



(i) To find the time in flight:

The height above the ground is given by:

$s_y = s_0 + usin\theta . t + \frac{1}{2}(-g)t^2$

So we need to solve $20 + (-10)(0.5)t - \frac{1}{2}(9.8)t^2 = 0$

or $9.8t^2 + 10t - 40 = 0$

$\Rightarrow t = \frac{-10+\sqrt{100-4(9.8)(-40)}}{2(9.8)}$ (taking the +ve root)

$= 1.5735 = 1.57s\ (3sf)$

(ii) To find the speed on hitting the ground:

The vertical component of velocity is given by

${v_y}^2 = {u_y}^2 + 2(-g)s_y$,

where s_y is the displacement (in the vertical direction) relative to the starting point (ie $20m$ above the ground in this case)

[This can be seen by putting $s_y = 0$, which gives $v_y = u_y$; noting that the other solution of $v_y = -u_y$ is found when the trajectory is extended backwards (ie into negative time) until the same height is reached, where by symmetry the vertical speed will be the same, but upwards instead of downwards - bearing in mind that u_y is negative at the starting point, so that $-u_y$ gives upward motion.]

$\Rightarrow {v_y}^2 = (-10sin30°)^2 + 2(-9.8)(-20)$

$= 25 + 392 = 417$

$\Rightarrow v_y = -20.421$

$v_x = 10cos30° = 8.6603$

Hence speed $= \sqrt{(8.6603)^2 + (-20.421)^2}$

$= 22.181 = 22.2ms^{-1}(3sf)$

[Note: Had we been starting at ground level, then by symmetry:

$v_y = -u_y$]

(6) Example

A golf ball is hit with speed u at an angle of θ to the ground, which is horizontal.

The range of the ball will be $R = \frac{sin2\theta}{g}u^2$ and so there will be two values of θ (θ_1 & θ_2 , where $\theta_1 < \theta_2$) for which this range is attained. The time in flight will be $\frac{2usin\theta}{g}$, and let $T_1 = \frac{2usin\theta_1}{g}$ and $T_2 = \frac{2usin\theta_2}{g}$. Then $T_2 > T_1$, and $\frac{T_2}{T_1} = \frac{sin\theta_2}{sin\theta_1}$.

Thus, if $\theta = 30°$ and $R = 200m$ (with $g = 9.8$), $u = \sqrt{\frac{9.8(200)}{\left(\frac{\sqrt{3}}{2}\right)}} =$

$47.6ms^{-1}$ (approx. $47.6 \times 2.25 \approx 107$mph).

And the same range could have been achieved with

$2\theta = 180 - 2(30)$; ie $\theta = 60°$.

And $\frac{T_2}{T_1} = \frac{\sin(60°)}{\sin(30°)} = \frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)} = \sqrt{3}$; ie the higher trajectory takes about 1.7 times as long as the lower one.

Exercise Show that $\frac{T_2}{T_1} = \frac{1+\sqrt{1-k^2}}{k}$, where $k = \frac{gR}{u^2}$

Solution

Let $T = \frac{2usin\theta}{g}$ be the time in flight, and $R = \frac{sin2\theta}{g}u^2$ be the range.

Then $sin^2\theta = \left(\frac{gT}{2u}\right)^2$,

and so $sin^2(2\theta) = 4sin^2\theta(1 - sin^2\theta) = 4\left(\frac{gT}{2u}\right)^2(1 - \left(\frac{gT}{2u}\right)^2)$

Also $sin^2(2\theta) = \left(\frac{gR}{u^2}\right)^2$,

so that $4\left(\frac{gT}{2u}\right)^2\left(1 - \left(\frac{gT}{2u}\right)^2\right) = \left(\frac{gR}{u^2}\right)^2$

$\Rightarrow T^2\left(u^2 - \frac{1}{4}g^2T^2\right) = R^2$

ie $g^2T^4 - 4u^2T^2 + 4R^2 = 0$

$\Rightarrow T^2 = \frac{4u^2 \pm \sqrt{16u^4 - 16g^2R^2}}{2g^2}$

Then, writing $k = \frac{gR}{u^2}$ (so that $k = sin2\theta$),

$(\frac{T_2}{T_1})^2 = \frac{1+\sqrt{1-k^2}}{1-\sqrt{1-k^2}} = \frac{\left(1+\sqrt{1-k^2}\right)^2}{\left(1-\sqrt{1-k^2}\right)\left(1+\sqrt{1-k^2}\right)}$

$\Rightarrow \frac{T_2}{T_1} = \frac{1+\sqrt{1-k^2}}{\sqrt{1-(1-k^2)}} = \frac{1+\sqrt{1-k^2}}{k}$ (as $0 < k < 1$)

[Check: For the previous example, $k = \frac{gR}{u^2} = sin2\theta = \frac{\sqrt{3}}{2}$,

and $\frac{1+\sqrt{1-k^2}}{k} = \frac{1+\sqrt{1-\frac{3}{4}}}{(\frac{\sqrt{3}}{2})} = \sqrt{3}$, as before.]

(7) Summary of methods

(i) To find the maximum height reached:

(a) $v_y = 0$, using $v_y{}^2 = u_y{}^2 + 2(-g)s_y$

[noting that s_y is measured from the starting point, rather than ground level]

(b) Using other suvat equations, if the time taken to reach the maximum height has already been found.

(ii) To find the time taken to reach the maximum height:

(a) $v_y = 0$, using $v_y = u_y + (-g)t$

(b) Using other suvat equations, if the maximum height has already been found.

(iii) To find the time in flight:

(a) If starting at ground level, double the time taken to reach the maximum height.

(b) Alternatively, if starting at ground level:

By symmetry, $v_y = -u_y$,

so that $v_y = u_y + (-g)t \Rightarrow t = \frac{2u_y}{g}$

(c) Otherwise: $s_0 + u_y t + \frac{1}{2}(-g)t^2 = 0$

(iv) To find the range:

(a) If T is the time in flight: range $= u_x T$

(b) Setting $y = 0$ in the cartesian equation:

$xtan\theta - \frac{gx^2 sec^2\theta}{2u^2} = 0$

(v) To find the vertical component of the velocity on hitting the ground:

(a) If starting at ground level, $v_y = -u_y$

(b) From ${v_y}^2 = {u_y}^2 + 2(-g)s_y$

[noting that s_y is measured from the starting point, rather than ground level]

(c) From $v_y = u_y + (-g)T$, where T is the time in flight

(8) General Formulae

(i) For the maximum height (H),

'$v^2 = u^2 + 2as' \Rightarrow 0 = (usin\theta)^2 + 2(-g)H$

$\Rightarrow H = \frac{u^2 sin^2\theta}{2g}$

(ii) For the time to reach the maximum height (T),

$'v = u + at' \Rightarrow 0 = usin\theta - gT$

$\Rightarrow T = \frac{usin\theta}{g}$

(iii) From (3)(iii), the range is $\frac{sin2\theta.u^2}{g}$

Summary

maximum height	$\frac{u^2 sin^2\theta}{2g}$
time to reach maximum height	$\frac{usin\theta}{g}$
range	$\frac{sin2\theta.u^2}{g}$