

Polar Curves Exercises – with Sol'ns (15 pages; 3/2/26)

(1)(i) Sketch the curve $r = 5 + 4\cos\theta$, explaining how you obtained your sketch. [Do not obtain your curve entirely by plotting points.]

(ii) Without converting the curve to cartesian form, find the greatest negative x -coordinate of a point on the curve.

(iii) Determine the area enclosed by the curve.

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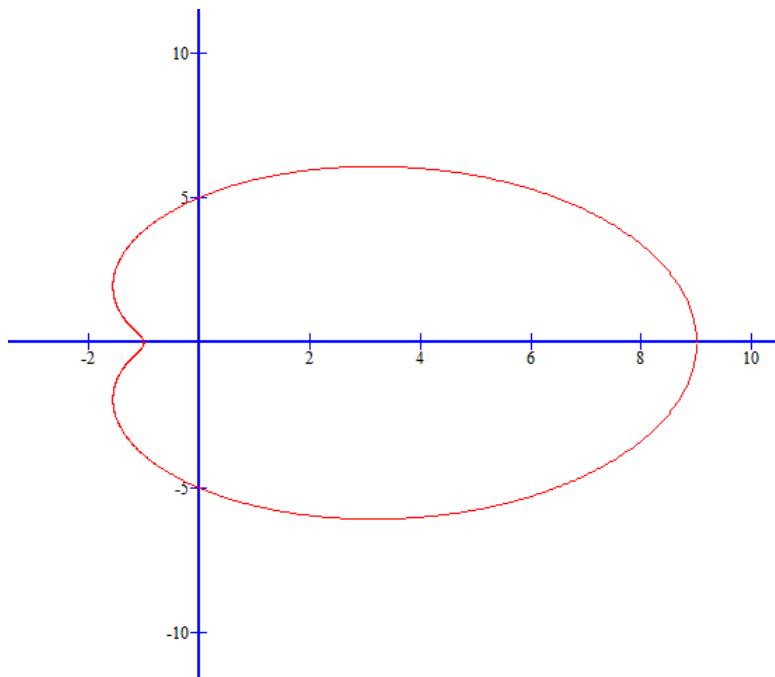
Solution

(i) $r = 5 + 4\cos\theta$

Step 1: As r is a function of $\cos\theta$, the curve will be symmetric about the x -axis.

Step 2: $r > 0$ at all times

Step 3: Key points to plot are $\theta = 0, r = 9$; $\theta = \frac{\pi}{2}, r = 5$; $\theta = \pi, r = 1$



(ii) The required x -coordinate can be found by investigating the vertical tangents; ie when $\frac{dx}{d\theta} = 0$ [when the x -coordinate is (instantaneously) not changing as θ changes]

$$x = r \cos \theta = (5 + 4 \cos \theta) \cos \theta$$

$$\text{so that } \frac{dx}{d\theta} = (-4 \sin \theta) \cos \theta + (5 + 4 \cos \theta)(-\sin \theta) = -8 \sin \theta \cos \theta - 5 \sin \theta$$

$$\text{Then } \frac{dx}{d\theta} = 0 \Rightarrow \sin \theta = 0 \text{ (ie } \theta = 0 \text{ or } \pi) \text{ or } \cos \theta = -\frac{5}{8}$$

$$\Rightarrow x = (5 + 4 \cos \theta) \cos \theta = \left(5 - \frac{20}{8}\right) \left(-\frac{5}{8}\right) = -\frac{25}{16}$$

$$\text{(iii) Area enclosed by curve} = 2 \int_0^{\pi} \frac{1}{2} (5 + 4 \cos \theta)^2 d\theta$$

$$= \int_0^{\pi} 25 + 16 \cos^2 \theta + 40 \cos \theta d\theta$$

$$= \int_0^{\pi} 25 + 8(1 + \cos 2\theta) + 40 \cos \theta d\theta$$

$$= [33\theta + 4 \sin 2\theta + 40 \sin \theta]_0^{\pi}$$

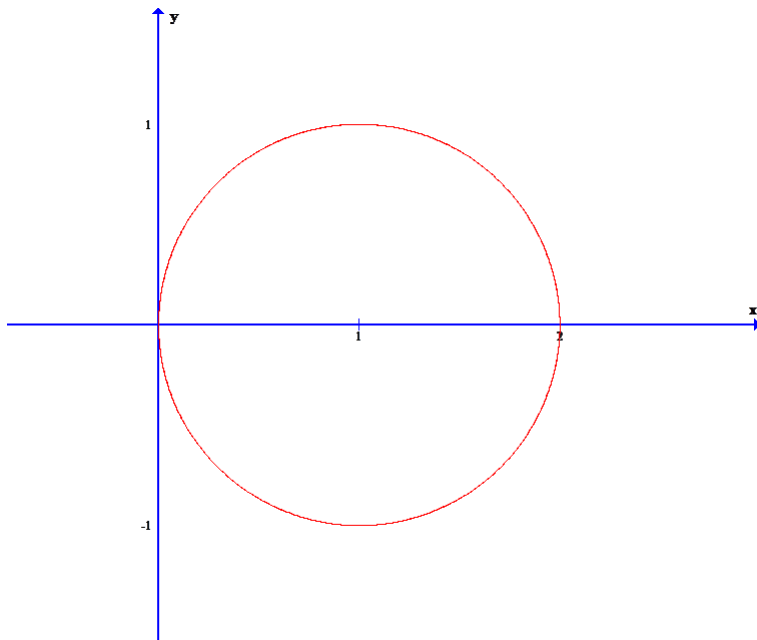
$$= 33\pi$$

[Rough check: Area of rectangle of base 11 and height 10 is approx. 35π]

(2) Convert the curve $(x - 1)^2 + y^2 = 1$ to polar form.

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Solution



$$x = r\cos\theta \text{ and } y = r\sin\theta$$

$$\text{So } r^2\cos^2\theta + 1 - 2r\cos\theta + r^2\sin^2\theta = 1$$

$$\Rightarrow r^2 - 2r\cos\theta = 0$$

$$\Rightarrow r = 2\cos\theta \text{ or } r = 0$$

$$\text{ie } r = 2\cos\theta$$

$$[\text{with } r = 0 \text{ when } \theta = \frac{\pi}{2}]$$

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Solution

$$r = \frac{2}{1+\cos\theta}; x = r\cos\theta \text{ and } y = r\sin\theta$$

$$\text{Also } r^2 = x^2 + y^2$$

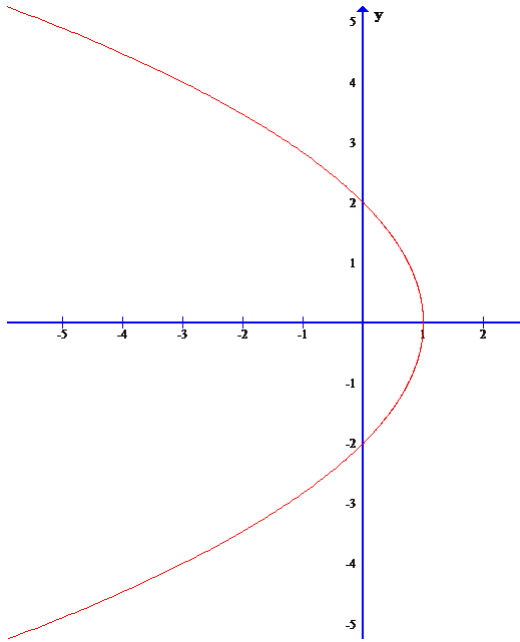
$$\text{So } r + r\cos\theta = 2 \Rightarrow r = 2 - x \Rightarrow r^2 = (2 - x)^2$$

$$\Rightarrow x^2 + y^2 = 4 + x^2 - 4x \Rightarrow y^2 = 4(1 - x)$$

This can be obtained from the parabola $y^2 = 4x$ by the following steps:

$y^2 = 4(-x) = -4x$ [reflection in the y -axis; note that the curve now only exists for negative x]

$$y^2 = -4(x - 1) = 4(1 - x) \text{ [translation of } \begin{pmatrix} 1 \\ 0 \end{pmatrix}]$$



(4)(i) Sketch the curve $r^2 = \sin 2\theta$.

(ii) Show how to sketch the curve $r^2 = \cos 2\theta$ by applying a transformation to $r^2 = \sin 2\theta$.

(iii) Find the largest y -coordinate of the curve $r^2 = \sin 2\theta$.

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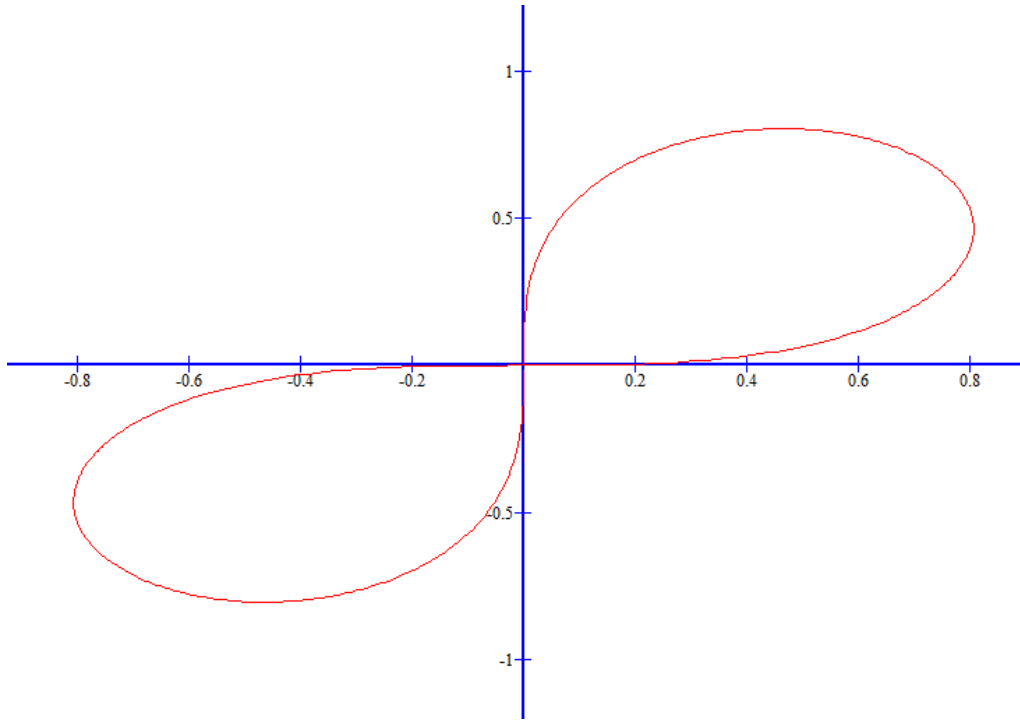
(i) Step 1: As $r = \pm\sqrt{\sin 2\theta}$ isn't a function of either $\cos\theta$ or $\sin\theta$, there is no symmetry about the x or y axis.

Step 2: The curve isn't defined for $\frac{\pi}{2} < \theta < \pi$ or for $\frac{3\pi}{2} < \theta < 2\pi$ (as $\sin 2\theta < 0$).

Step 3: For each θ there will positive and negative values of r of the same magnitude. [However the negative values of r for θ will overlap with the positive values for $\theta + \pi$.]

Step 4: Key points to plot are: $\theta = 0, r = 0$; $\theta = \frac{\pi}{4}, r = \pm 1$; $\theta = \frac{\pi}{2}, r = 0$ (and the cycle repeats itself for $\theta = \pi$ to $\theta = \frac{3\pi}{2}$).

Step 5: The gradient at $\theta = 0$ (when $r = 0$) is 0 (ie along the line $\theta = 0$), and at $\theta = \frac{\pi}{2}$ it is ∞ (ie along the line $\theta = \frac{\pi}{2}$).



(ii) $r = 1$ when $\theta = \frac{\pi}{4}$ for $r^2 = \sin 2\theta$, and when $\theta = 0$ for $r^2 = \cos 2\theta$, so the curve for $r^2 = \sin 2\theta$ needs to be rotated by $\frac{\pi}{4}$ clockwise.

[This rotation transforms $r^2 = \sin 2\theta$ to $r^2 = \sin 2(\theta + \frac{\pi}{4})$ [as clockwise is the negative direction] $= \sin(2\theta + \frac{\pi}{2}) = \cos 2\theta$]

$$(iii) y = r \sin \theta, r^2 = \sin 2\theta \Rightarrow y^2 = \sin 2\theta \cdot \sin^2 \theta$$

$$\Rightarrow 2y \frac{dy}{d\theta} = 2 \cos 2\theta \sin^2 \theta + \sin 2\theta (2 \sin \theta \cos \theta)$$

$$\text{Then } \frac{dy}{d\theta} = 0 \Rightarrow (\cos^2 \theta - \sin^2 \theta) \sin^2 \theta + (2 \sin \theta \cos \theta)(\sin \theta \cos \theta) = 0$$

$$\Rightarrow 3 \cos^2 \theta \sin^2 \theta - \sin^4 \theta = 0$$

$$\Rightarrow \sin^2 \theta (3 \cos^2 \theta - \sin^2 \theta) = 0$$

$$\Rightarrow \sin^2 \theta (3 \cos^2 \theta - (1 - \cos^2 \theta)) = 0$$

$$\Rightarrow \sin^2 \theta (4 \cos^2 \theta - 1) = 0$$

$$\Rightarrow \theta = 0 \text{ or } \pi \text{ (within } [0, 2\pi)) \text{ (ie when the curve is at the Origin)}$$

$$\text{or } \cos \theta = \pm \frac{1}{2}, \text{ so that } \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3} \text{ or } \frac{5\pi}{3}$$

The curve doesn't exist for $\theta = \frac{2\pi}{3}$ and $\frac{5\pi}{3}$, and so the required value is

$$\theta = \frac{\pi}{3} \text{ (when the } y\text{-coordinate is positive).}$$

$$\text{At } \theta = \frac{\pi}{3}, y^2 = \sin 2\theta \cdot \sin^2 \theta = \left(\frac{\sqrt{3}}{2}\right) \left(\frac{\sqrt{3}}{2}\right)^2 = \left(\frac{\sqrt{3}}{2}\right)^3$$

(5) A curve has polar equation $r = 3(\cos\theta + 2\sin\theta)$,
for $0 \leq \theta \leq \pi$.

(i) Show that the curve is a circle.

(ii) Determine the polar coordinates of the point on the curve
which is furthest from the pole.

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Solution

$$(i) \ r = 3(\cos\theta + 2\sin\theta) \Rightarrow r^2 = 3r\cos\theta + 6r\sin\theta,$$

$$\Rightarrow x^2 + y^2 = 3x + 6y$$

$$\Rightarrow (x - \frac{3}{2})^2 + (y - 3)^2 - \frac{3^2}{4} - 3^2 = 0$$

$$\Rightarrow (x - \frac{3}{2})^2 + (y - 3)^2 = \frac{5(9)}{4}; \text{ ie a circle with radius } \frac{3\sqrt{5}}{2}$$

$$(ii) \ \frac{dr}{d\theta} = 3(-\sin\theta + 2\cos\theta)$$

$$\frac{dr}{d\theta} = 0 \Rightarrow \tan\theta = 2 \text{ (as } \cos\theta \neq 0)$$

$$\frac{d^2r}{d\theta^2} = 3(-\cos\theta - 2\sin\theta) = -3\cos\theta(1 + 2\tan\theta) \text{ (when } \cos\theta \neq 0)$$

So when $\tan\theta = 2$, $\frac{d^2r}{d\theta^2} < 0$, as $0 < \theta < \frac{\pi}{2}$, so that $\cos\theta > 0$,

and hence r is a maximum when $\tan\theta = 2$

$$\text{When } \tan\theta = 2, \sec^2\theta = 2^2 + 1 = 5,$$

$$\text{so that } \cos\theta = \sqrt{\frac{1}{5}} \text{ (as } \cos\theta > 0)$$

$$\text{and } r = 3\cos\theta(1 + 2\tan\theta) = \frac{3}{\sqrt{5}}(1 + 4) = 3\sqrt{5}$$

The polar coordinates are thus $(3\sqrt{5}, \tan^{-1}2)$.