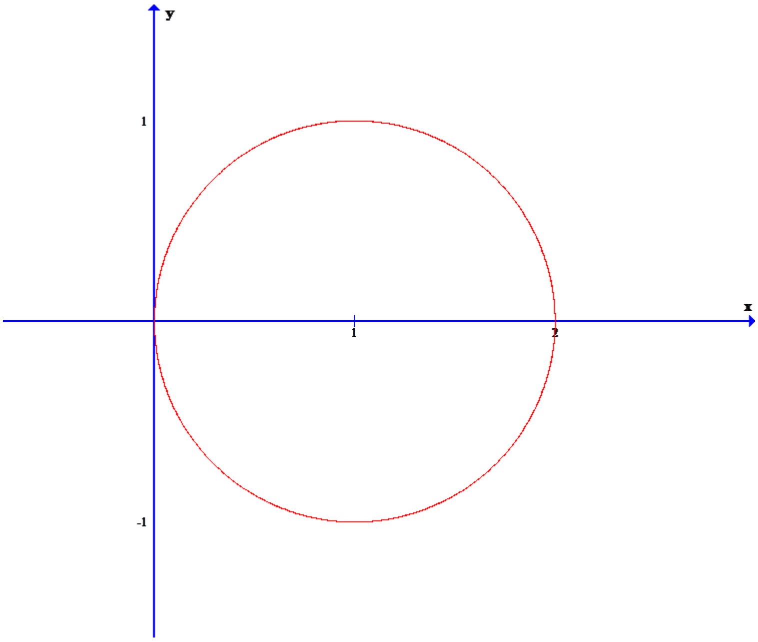
Polar Curves - Exercises (Sol'ns) (6 pages; 14/01/20)

(1**) Convert the curve $(x-1)^2 + y^2 = 1$ to polar form.

Solution



$x = rcos\theta$ and $y = rsin\theta$

So $r^2cos^2\theta + 1 - 2rcos\theta + r^2sin^2\theta = 1$

$\Rightarrow r^2 - 2rcos\theta = 0$

$\Rightarrow r = 2cos\theta$ or $r = 0$

ie $r = 2cos\theta$ [with $r = 0$ when $\theta = \frac{\pi}{2}$]

(2***) Convert the curve $r = \frac{2}{1+cos\theta}$ to cartesian form, and sketch the curve, based on its cartesian form.

Solution

$r = \frac{2}{1+cos\theta}$; $x = rcos\theta$ and $y = rsin\theta$; also $r^2 = x^2 + y^2$

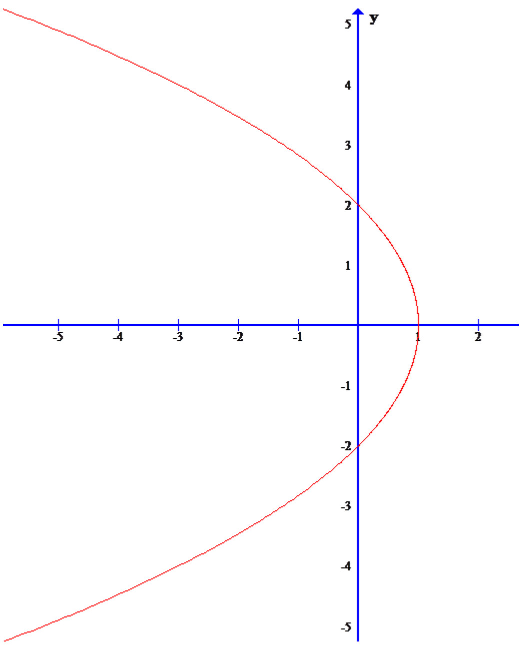
So $r + rcos\theta = 2 \Rightarrow r = 2 - x \Rightarrow r^2 = (2 - x)^2$

$\Rightarrow x^2 + y^2 = 4 + x^2 - 4x \Rightarrow y^2 = 4(1 - x)$

This can be obtained from the parabola $y^2 = 4x$ by the following steps:

$y^2 = 4(-x) = -4x$ [reflection in the y-axis; note that the curve now only exists for negative x]

$y^2 = -4(x - 1) = 4(1 - x)$ [translation of $\binom{1}{0}$]



(3***)(i) Sketch the curve $r = 5 + 4cos\theta$.

(ii) Without converting the curve to cartesian form, find the greatest negative x-coordinate of a point on the curve.

(iii) Determine the area enclosed by the curve.

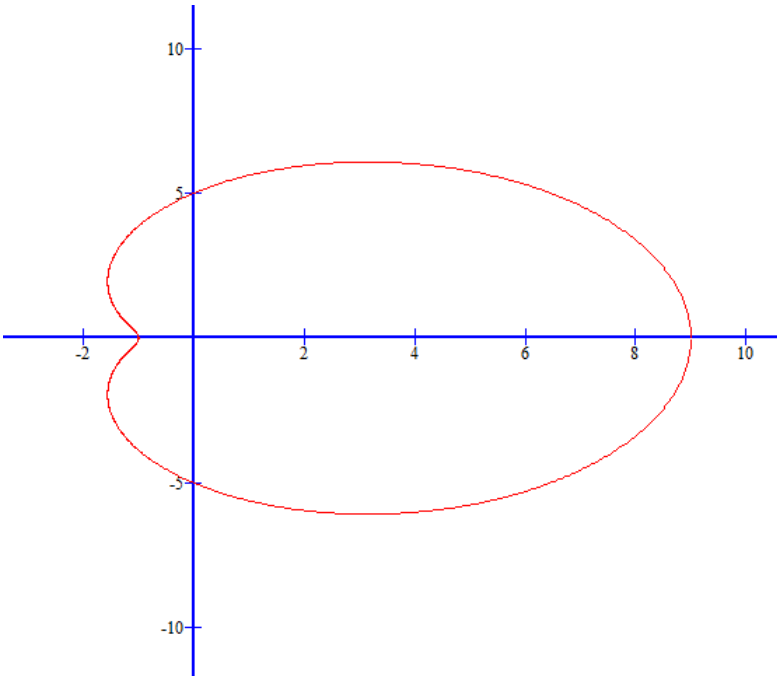
Solution

(i) $r = 5 + 4cos\theta$

Step 1: As r is a function of $cos\theta$, the curve will be symmetric about the x-axis.

Step 2: $r > 0$ at all times

Step 3: Key points to plot are $\theta = 0, r = 9;\ \theta = \frac{\pi}{2}, r = 5;\ \theta = \pi, r = 1$



(ii) The required x-coordinate can be found by investigating the vertical tangents; ie when $\frac{dx}{d\theta} = 0$ [when the x-coordinate is (instantaneously) not changing as θ changes]

$x = rcos\theta = (5 + 4\cos\theta)cos\theta$

so that $\frac{dx}{d\theta} = (-4sin\theta)cos\theta + (5 + 4cos\theta)(-sin\theta) =$
$-8sin\theta cos\theta - 5sin\theta$

Then $\frac{dx}{d\theta} = 0 \Rightarrow sin\theta = 0$ (ie $\theta = 0\ or\ \pi$) or $cos\theta = -\frac{5}{8}$

$\Rightarrow x = (5 + 4\cos\theta)cos\theta = \left(5 - \frac{20}{8}\right)\left(-\frac{5}{8}\right) = -\frac{25}{16}$

(iii) Area enclosed by curve $= 2\int_0^{\pi} \frac{1}{2}(5 + 4\cos\theta)^2 d\theta$

$= \int_0^{\pi} 25 + 16cos^2\theta + 40cos\theta\ d\theta$

$= \int_0^{\pi} 25 + 8(1 + cos2\theta) + 40cos\theta\ d\theta$

$= [33\theta + 4sin2\theta + 40sin\theta]_0^{\pi}$

$= 33\pi$

[Rough check: Area of rectangle of base 11 and height 10 is approx. 35π]

(4***)(i) Sketch the curve $r^2 = sin2\theta$.

(ii) Show how to sketch the curve $r^2 = cos2\theta$ by applying a transformation to $r^2 = sin2\theta$.

(iii) Find the largest y-coordinate of the curve $r^2 = sin2\theta$.

Solution

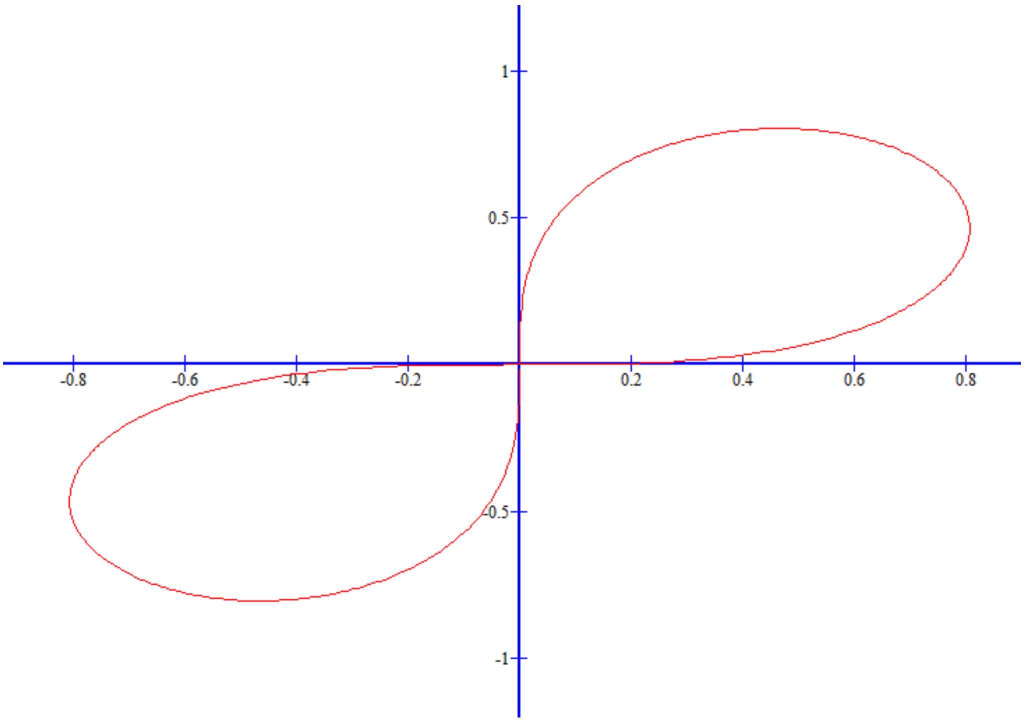
(i) Step 1: As $r = \pm\sqrt{sin2\theta}$ isn't a function of either $cos\theta$ or $sin\theta$, there is no symmetry about the x or y axis.

Step 2: The curve isn't defined for $\frac{\pi}{2} < \theta < \pi$ or for $\frac{3\pi}{2} < \theta < 2\pi$ (as $sin2\theta < 0$).

Step 3: For each θ there will positive and negative values of r of the same magnitude. [However the negative values of r for θ will overlap with the positive values for $\theta + \pi$.]

Step 4: Key points to plot are: $\theta = 0, r = 0$; $\theta = \frac{\pi}{4}, r = \pm 1$; $\theta = \frac{\pi}{2}, r = 0$ (and the cycle repeats itself for $\theta = \pi$ to $\theta = \frac{3\pi}{2}$).

Step 5: The gradient at $\theta = 0$ (when $r = 0$) is 0 (ie along the line $\theta = 0$), and at $\theta = \frac{\pi}{2}$ it is ∞ (ie along the line $\theta = \frac{\pi}{2}$).



(ii) $r = 1$ when $\theta = \frac{\pi}{4}$ for $r^2 = sin2\theta$, and when $\theta = 0$ for $r^2 = cos2\theta$, so the curve for $r^2 = sin2\theta$ needs to be rotated by $\frac{\pi}{4}$ clockwise.

[This rotation transforms $r^2 = sin2\theta$ to $r^2 = sin2(\theta + \frac{\pi}{4})$ [as clockwise is the negative direction] $= \sin\left(2\theta + \frac{\pi}{2}\right) = cos2\theta$]

(iii) $y = rsin\theta, r^2 = sin2\theta \Rightarrow y^2 = sin2\theta. sin^2\theta$

$\Rightarrow 2y\frac{dy}{d\theta} = 2cos2\theta sin^2\theta + sin2\theta(2sin\theta cos\theta)$

Then $\frac{dy}{d\theta} = 0 \Rightarrow (cos^2\theta - sin^2\theta)sin^2\theta + (2sin\theta cos\theta)(sin\theta cos\theta) = 0$

$\Rightarrow 3cos^2\theta sin^2\theta - sin^4\theta = 0$

$\Rightarrow sin^2\theta(3cos^2\theta - sin^2\theta) = 0$

$\Rightarrow sin^2\theta(3cos^2\theta - (1 - cos^2\theta)) = 0$

$\Rightarrow sin^2\theta(4cos^2\theta - 1) = 0$

$\Rightarrow \theta = 0 \; or \; \pi$ (within $[0, 2\pi)$) (ie when the curve is at the Origin)

or $cos\theta = \pm\frac{1}{2}$, so that $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3} \; or \; \frac{5\pi}{3}$

The curve doesn't exist for $\theta = \frac{2\pi}{3} \; and \; \frac{5\pi}{3}$, and so the required value is

$\theta = \frac{\pi}{3}$ (when the y-coordinate is positive).

At $\theta = \frac{\pi}{3}, y^2 = sin2\theta. sin^2\theta = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)^2 = \left(\frac{\sqrt{3}}{2}\right)^3$