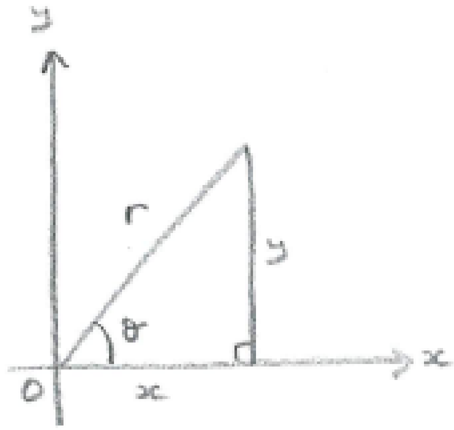
Polar Curves (21 pages; 2/11/20)

[See also Important Ideas: Polar Curves, for a summary of the main points.]

(1) Obtaining polar coordinates (r, θ) from Cartesian coordinates



$r^2 = x^2 + y^2$

$tan\theta = \frac{y}{x}$

Either $0 \leq \theta < 2\pi$ ['2π convention']

or $-\pi < \theta \leq \pi$ ['π convention']

θ is undefined for the Origin

A negative r can be catered for as follows: If θ is $\frac{\pi}{4}$ and $r = -1$, for example, then the point would be plotted at $\left(\frac{\pi}{4} + \pi, 1\right)$.

Some textbooks (and exam boards), however, exclude the parts of the curve where $r < 0$. Sometimes these parts are indicated by a dotted curve. However, even if the curve isn't to be drawn for

$r < 0$, it helps the curve-sketching process to be aware of where this dotted curve would be.

Note that, for example, the points $(1, 1)$ & $(-1, -1)$ have the same value of $tan\theta$ (since $\tan(\frac{5\pi}{4}) = tan\left(\frac{\pi}{4}\right)$). We therefore have to consider which quadrant the point is in.

It is possible to use polar graph paper, which has concentric circles for particular values of r , and radial lines at $\theta = \frac{\pi}{6}$, $\frac{\pi}{3}$ $etc.$

(2) Obtaining Cartesian coordinates from polar coordinates

$x = rcos\theta$; $y = rsin\theta$

(3) Sketching graphs

Example: $r = 1 + sin\theta$ (see graph)

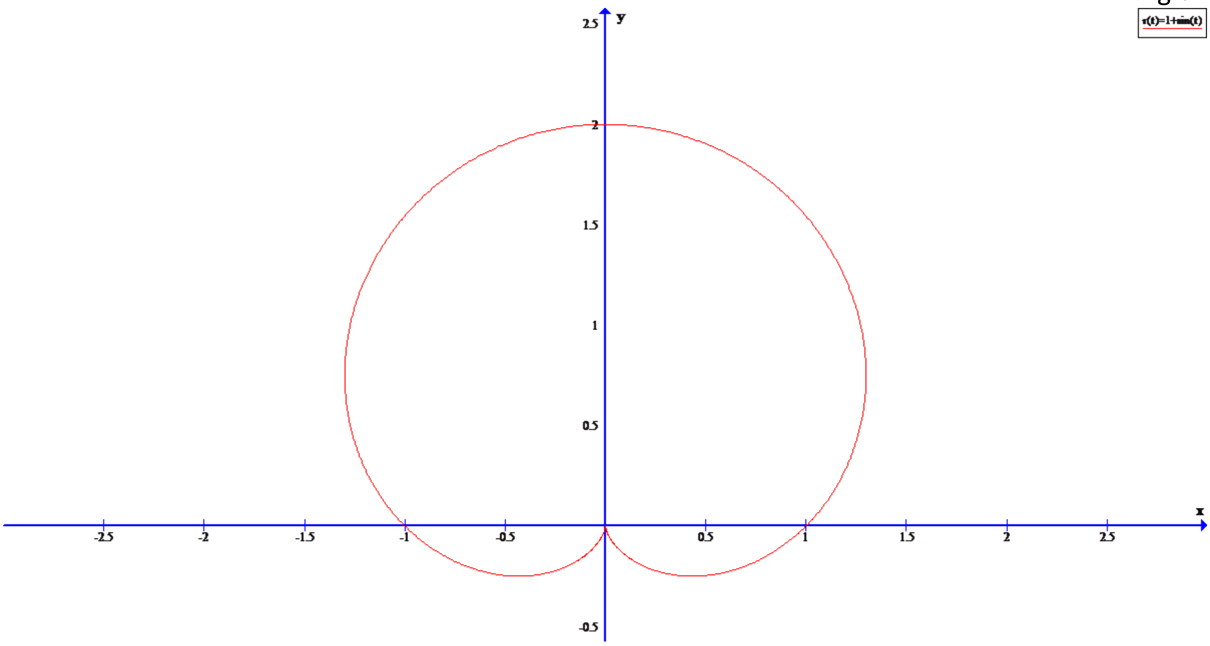
(i) Plot points for convenient values of θ, such as 0 & $\frac{\pi}{2}$

(ii) Any function of $sin\theta$ will be symmetric about the y axis (ie $\theta = \frac{\pi}{2}$) [The reflection of a general function $y = f(\theta)$ about $\theta = \frac{\pi}{2}$ is $y = f(\pi - \theta)$, and $r(\pi - \theta) = r(\theta)$ for $r(\theta) = 1 + sin\theta$ (or any other function of $sin\theta$)]

(iii) As $\theta \to \frac{3\pi}{2}$, $r \longrightarrow 0$, and the direction of the curve approaches

$\theta = \frac{3\pi}{2}$ as $r \longrightarrow 0$

Note that $r < 0$ is not possible.



This shape is known as the 'cardioid'.

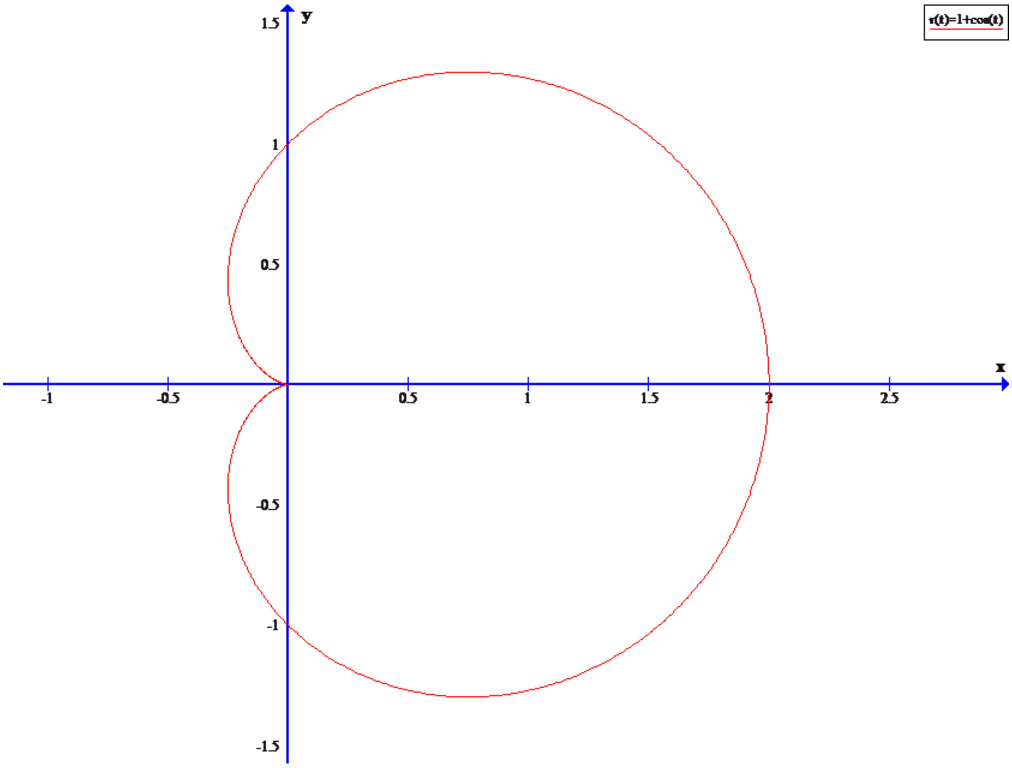
(4) **Example**: $r = 1 + cos\theta$ (see graph)

(i) Points can be plotted for $\theta = 0$ & $\theta = \frac{\pi}{2}$, as before.

(ii) As $\cos(-\theta) = cos\theta$, $r(-\theta) = r(\theta)$

and in general, any function of $cos\theta$ will be symmetric about the x axis (ie $\theta = 0$)

(iii) $r = 0 \Rightarrow \theta = \pi$



Note that, for example, $1 + cos0 = 1 + sin\left(\frac{\pi}{2}\right)$, so that $\theta = 0$ for $r = 1 + cos\theta$ corresponds to $\theta = \frac{\pi}{2}$ for $\mathrm{r} = 1 + sin\theta$, and the graph of $r = 1 + cos\theta$ can be obtained from that of $\mathrm{r} = 1 + sin\theta$ by a rotation of $\frac{\pi}{2}$ clockwise.

(5) In general, $r(2\alpha - \theta) = r(\theta) \Rightarrow$ symmetry about $\theta = \alpha$

(6) **Example**: $r = 1 + sin^2\theta$ (see graph)

(i) Points can be plotted for $\theta = 0$ & $\theta = \frac{\pi}{2}$, as before.

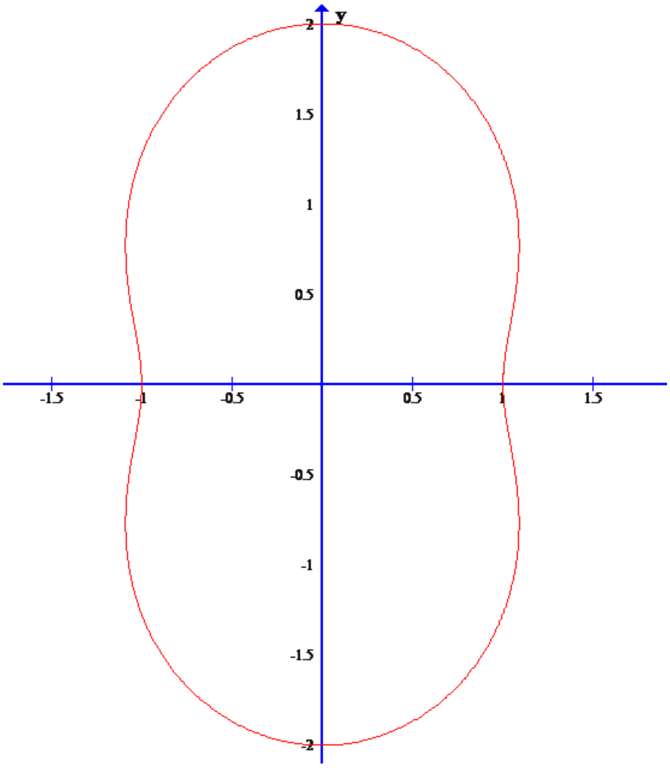
(ii) Being a function of $sin\theta$, there will be symmetry about the y axis. And as $sin^2\theta = 1 - cos^2\theta$, r is also a function of $cos\theta$, and hence there is symmetry about the x axis.

(iii) $r > 0$ for all θ

(iv) We could investigate the stationary points of r, as follows:

$\frac{dr}{d\theta} = 2sin\theta cos\theta = \sin(2\theta)$ and $\frac{d^2r}{d\theta^2} = 2\cos(2\theta)$,

At $\theta = 0$ & π, $\frac{dr}{d\theta} = 0$ and $\frac{d^2r}{d\theta^2} > 0$, so that there is a minimum for r. At $\theta = \frac{\pi}{2}$ & $\frac{3\pi}{2}$, $\frac{dr}{d\theta} = 0$ and $\frac{d^2r}{d\theta^2} < 0$, so that there is a maximum for r.



(7) **Example**: $r = cos2\theta$ (see graph)

(i) Points can be plotted for $\theta = 0$ & $\theta = \frac{\pi}{2}$

(ii) As $cos2\theta = cos^2\theta - sin^2\theta$, which can be expressed as either a function of $cos\theta$ or a function of $sin\theta$, there is symmetry about both $\theta = 0$ & $\theta = \frac{\pi}{2}$.

(iii) Graphs of the form $r = cos(n\theta)$ and $r = sin(n\theta)$ will consist of a number of petals, and the location of these petals can be established by considering their extremities, where $|r|$ takes its maximum value, in this case ± 1.

$r = 1 \Rightarrow \theta = 0, \pi$ (within the interval $[0,2\pi)$)

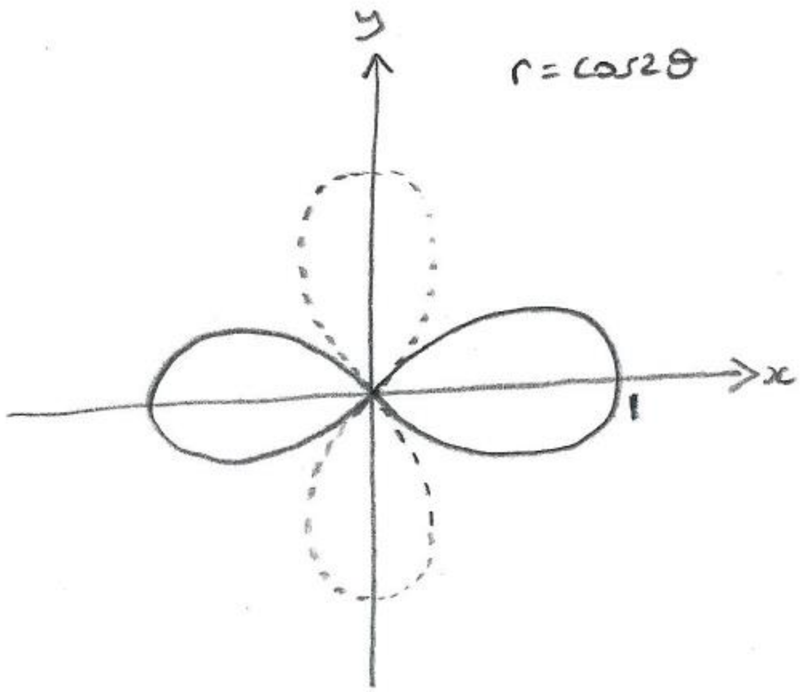
$r = -1 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

(iv) $r = 0 \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}$ ... (every $\frac{\pi}{2}$)

(iv) $r < 0$ when $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$ (lower petal of the graph) and

$\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$ (upper petal)

A useful device for drawing the parts of graphs where $r < 0$ is to imagine the hand of a clock sweeping round, but in an anti-clockwise direction. For $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$ in this example, when $r < 0$, the curve is on the far side of the Origin, relative to the clock hand, and as the hand continues round it is clear that the lower petal is being followed, rather than the lefthand one. Thus, by the time $\theta = \frac{\pi}{2}$ has been reached, the curve is at the bottom of the lower petal.



(8) **Example**: $r = sin2\theta$ (see graph)

(i) The same methods can be used as for $r = cos2\theta$.

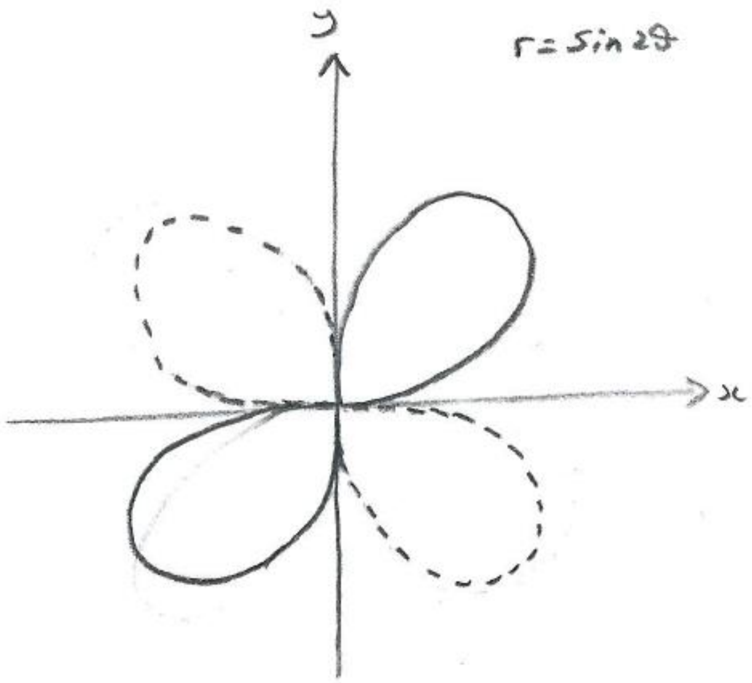
(ii) However, there is neither symmetry about the x or y axes (despite the appearance of the graph, as explained below):

$sin2\theta = 2sin\theta cos\theta$ is neither a function of $sin\theta$ or $cos\theta$, except for limited intervals, as $sin2\theta = 2sin\theta\sqrt{1 - sin^2\theta}$ for some values of θ (eg $\theta = \frac{\pi}{4}$), but $-2sin\theta\sqrt{1 - sin^2\theta}$ for others (eg $\theta = \frac{3\pi}{4}$) ; bearing in mind that $\sqrt{1 - sin^2\theta}$ means the positive square root. Thus, for $\theta = \frac{\pi}{4}$, $\sin 2\theta = 1$, but $sin2(\pi - \theta) = -1$, so that there isn't symmetry about the y axis.

There is apparent symmetry in the graph, but it should be noted that the dotted curve in the 2nd quadrant relates to an angle in the 4th quadrant (and vice versa), because $r < 0$.

(iii) $r = sin2\theta$ can also be obtained from $r = cos2\theta$ by a rotation, by noting that $r = sin2\theta$ is behind $r = cos2\theta$ by $\frac{\pi}{4}$

$[cos2(0) = sin2(\frac{\pi}{4})]$



Tip: To draw the graph of $r = \sin(n\theta)$, draw the graph of $r = cos(n\theta)$ (which has symmetry about the x-axis, where $\theta = 0$), and rotate it by $\frac{(\frac{\pi}{2})}{n}$ (anti-clockwise).

(9) **Example**: $r = sin3\theta$ (see graph)

(i) $r = 1 \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{9\pi}{6}$ (within the interval $[0,2\pi)$)

$r = -1 \Rightarrow \theta = \frac{3\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

Notice that $\frac{3\pi}{6} + \pi = \frac{9\pi}{6}, \frac{7\pi}{6} + \pi = \frac{13\pi}{6} = \frac{\pi}{6} + 2\pi$

and $\frac{11\pi}{6} + \pi = \frac{17\pi}{6} = \frac{5\pi}{6} + 2\pi$

This means that when the dotted curve is drawn at $r = -1$, it will overlap the bold curve at $r = 1$.

This is a feature of curves of the form $r = asin([2n+1]\theta)$

and $r = acos([2n+1]\theta)$, which will have $2n+1$ petals (each occurring twice).

Curves of the form $r = \text{asin}(2n\theta)$ and $r = \text{acos}(2n\theta)$ will have $2n$ bold petals and $2n$ dotted petals, as seen with $r = \sin(2\theta)$ and $r = \cos(2\theta)$.

(ii) It can be seen from the position of the petals ($\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$) that there is in fact symmetry about the y axis.

$sin3\theta$ can be shown to be expressible as $3sin\theta - 4sin^3\theta$

(eg using de Moivre's theorem), and is thus a function of $sin\theta$.

$sin([2n+1]\theta)$ can also be expressed in terms of powers of $sin\theta$, but $\sin(2n\theta)$ cannot.

(iii) Alternatively, to show that $r = sin3\theta$ has symmetry about the y axis, we can let $f(\theta) = sin3\theta$, and show that $f\left(\frac{\pi}{2} + \theta\right) = f(\frac{\pi}{2} - \theta)$ [or that $f(\pi - \theta) = f(\theta)$]

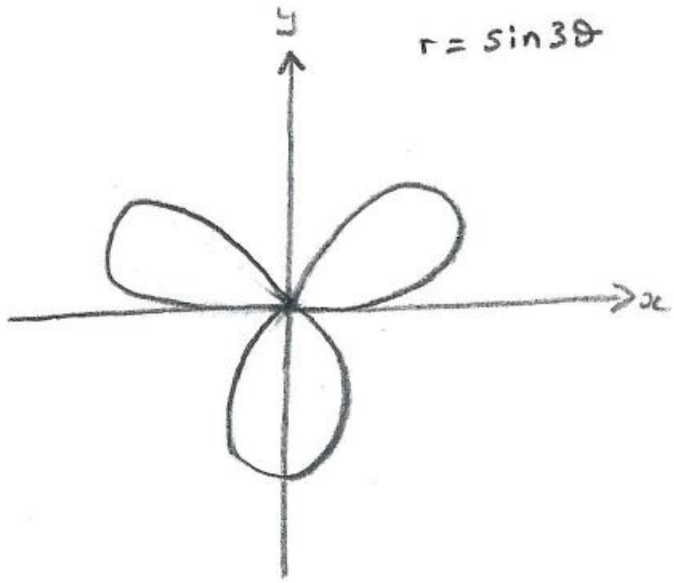
Thus:

$sin\left[3\left(\frac{\pi}{2} + \theta\right)\right] = sin\left(\frac{3\pi}{2} + 3\theta\right) = sin\left(3\theta - \frac{\pi}{2}\right) = -sin(\frac{\pi}{2} - 3\theta)$

$= -cos(3\theta)$

and $sin\left[3\left(\frac{\pi}{2} - \theta\right)\right] = sin\left(\frac{3\pi}{2} - 3\theta\right) = sin\left(-3\theta - \frac{\pi}{2}\right)$

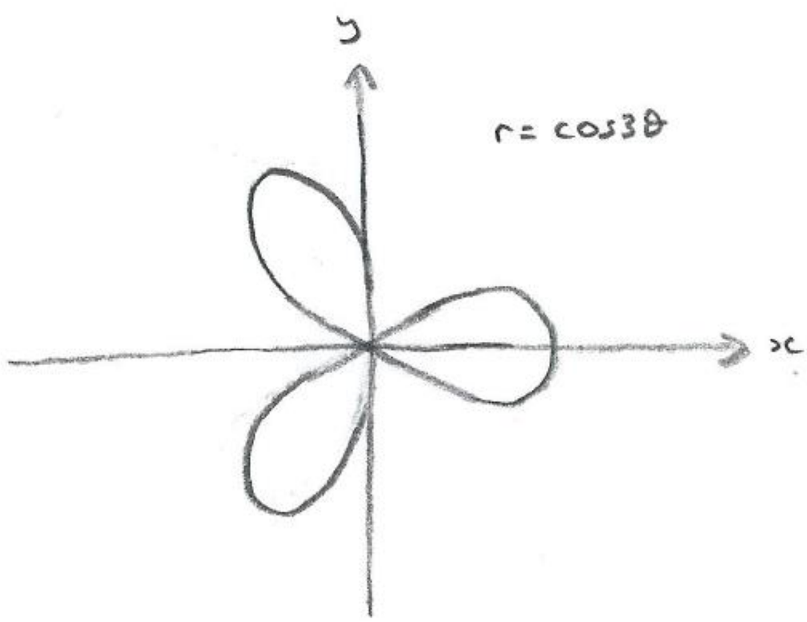
$= -sin(3\theta + \frac{\pi}{2}) = -sin\left(\frac{\pi}{2} - (-3\theta)\right) = -cos(-3\theta) = -cos(3\theta)$



(10) **Example**: $r = cos3\theta$ (see graph)

$r = cos\theta$ can be obtained from $r = sin\theta$ by rotating by $\frac{\pi}{2}$ radians clockwise, and we saw that $r = cos2\theta$ was obtained from $r = sin2\theta$ by a rotation of $\frac{\pi}{4}$ radians.

For $r = cos3\theta$, we rotate by $\frac{\frac{\pi}{2}}{3} = \frac{\pi}{6}$ radians.



(11) $\boldsymbol{r = \lambda + cos\theta}$

The following graphs show typical members of this family.

(The family is sometimes represented in the form

$r = a(p + qcos\theta)$, in which case $\frac{p}{q}$ has the role of λ. The presence of the a doesn't affect the shape of the curve.)

The values of λ can be classified as follows:

$\lambda = 0$: circle

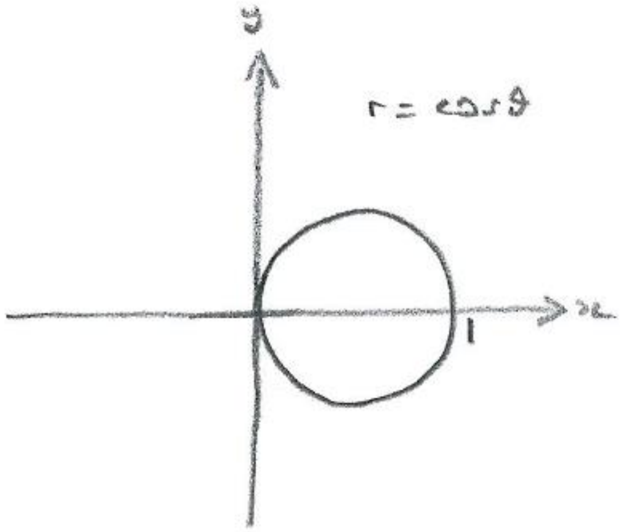
$0 < \lambda < 1$

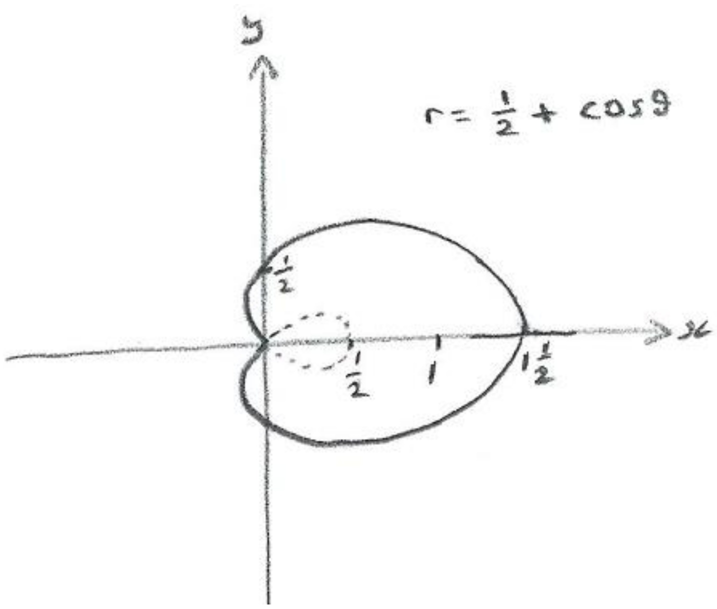
$\lambda = 1$: 'cardioid'

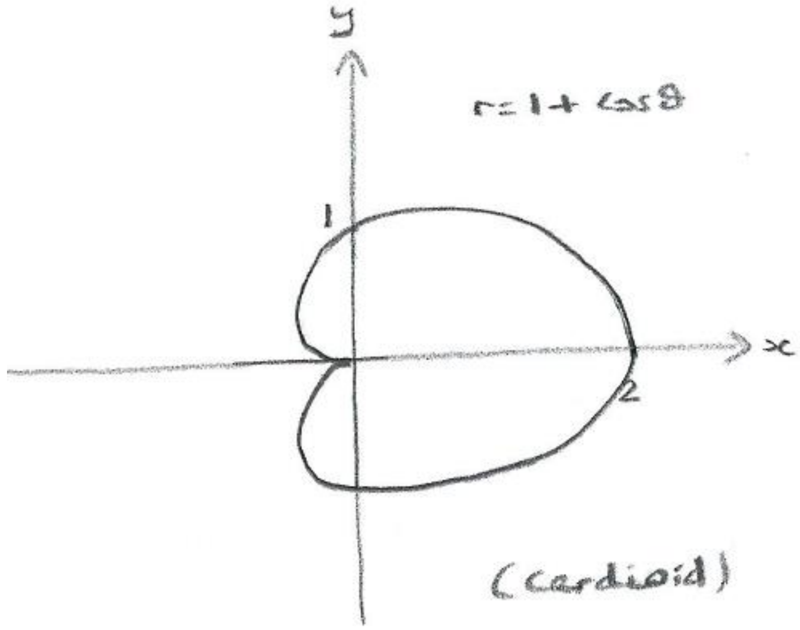
$1 < \lambda < 2$: 'dimple'

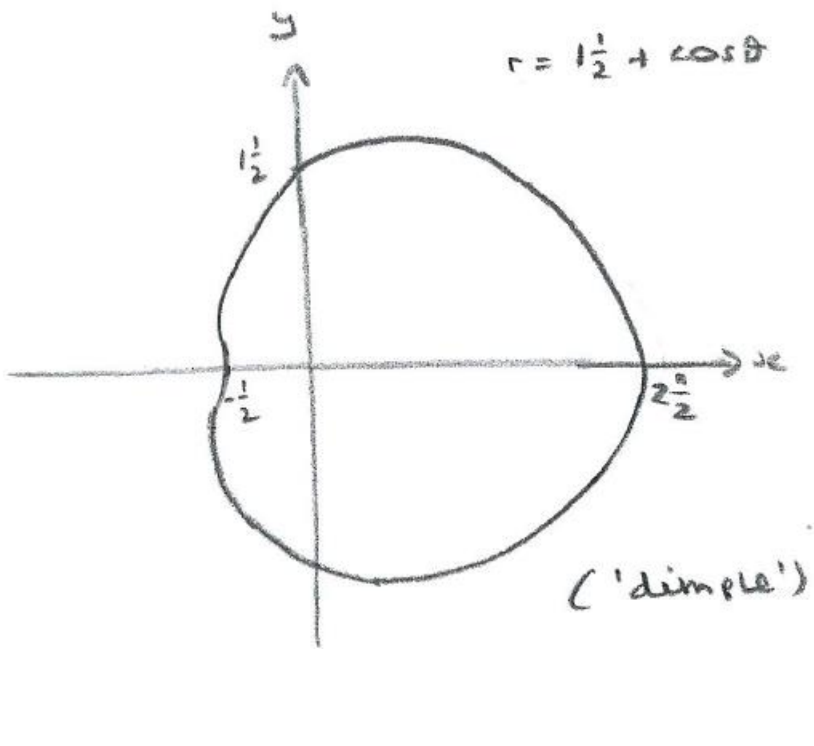
$\lambda \geq 2$: 'egg'

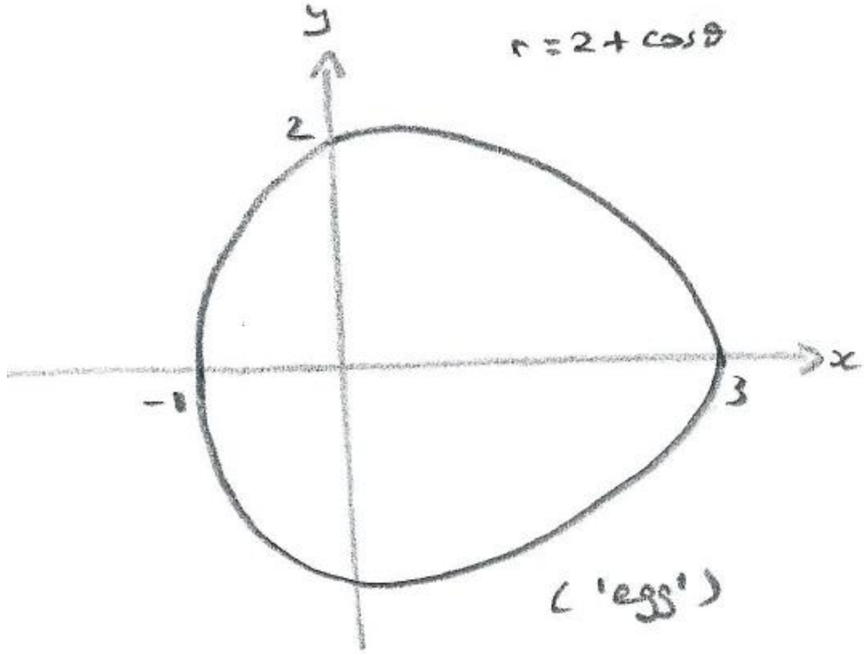
For $r = cos\theta$, note that (as discussed above) there is an odd number of petals (ie one), and the curve for $r < 0$ (when $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$) duplicates the curve for $r > 0$ (when $0 < \theta < \frac{\pi}{2}$ and $\frac{3\pi}{2} < \theta < 2\pi$).


y
r = cos θ
x
1


y
r = ½ + cos θ
½
x
½
1
1½


y
r = 1 + cos θ
1
x
2
(cardioid)

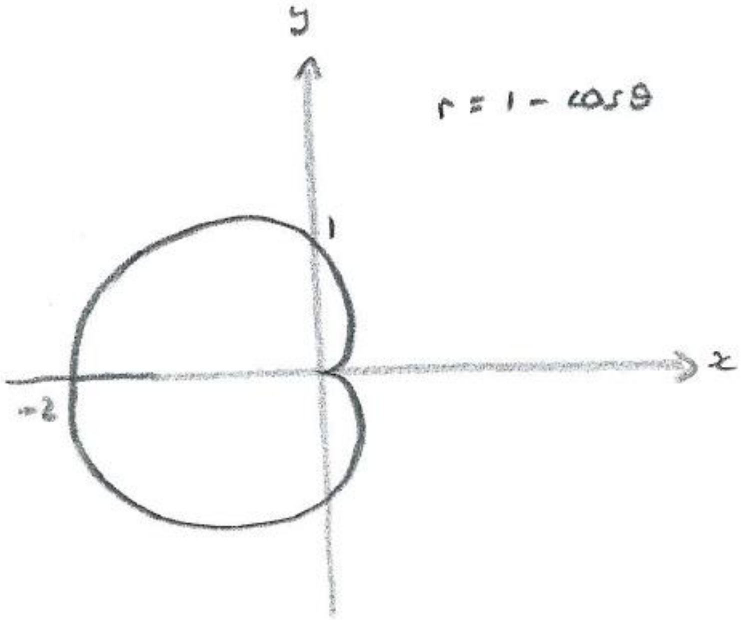

y
r = 1½ + cos θ
1½
x
-½
2½
('dimple')



(12) $\boldsymbol{r = 1 - cos\theta}$ (see graph)

$1 - cos\theta = 1 + cos(\pi - \theta)$, so the curve $r = 1 - cos\theta$ can be produced by working backwards from $\theta = \pi$ on the curve

$r = 1 + cos\theta$, to give a reflected cardioid

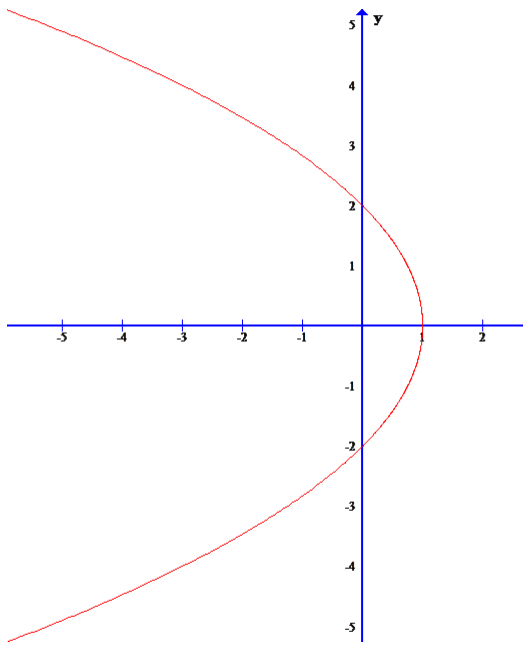


(13) **Example**: Convert the parabola $y^2 = 4(1-x)$ $(x \leq 1)$ to polar form.

Solution

First of all, to draw the curve in its Cartesian form, consider the series of transformations:

$y^2 = 4x \to y^2 \to 4(-x) \to 4(-[x-1]) = 4(1-x)$



$x = rcos\theta,\ \ y = rsin\theta$; hence $r^2sin^2\theta = 4(1 - rcos\theta)$

$\Rightarrow r^2sin^2\theta + 4rcos\theta - 4 = 0$

$\Rightarrow r = \frac{-4cos\theta \pm \sqrt{16cos^2\theta + 16sin^2\theta}}{2sin^2\theta}$

$= \frac{-2cos\theta \pm 2}{sin^2\theta} = \frac{2(1-cos\theta)}{(1-cos^2\theta)}$ or $\frac{-2(1+cos\theta)}{(1-cos^2\theta)}$

$= \frac{2}{1+cos\theta}$ or $\frac{-2}{1-cos\theta}$

For simplicity, we can require $r > 0$, so that $r = \frac{2}{1+cos\theta}$

[$\frac{-2}{1-cos\theta}$ can be written as $\frac{-2}{1+\cos(\pi-\theta)} = \frac{-2}{1+\cos(\theta-\pi)}$; thus the angle is changed by half a revolution, which balances the change of sign, to give the same point]

Also, stationary values of r can be investigated:

$\frac{dr}{d\theta} = -2(1 + cos\theta)^{-2}(-sin\theta)$

$\frac{dr}{d\theta} = 0 \Rightarrow\ \theta = 0$ (but not π) $\Rightarrow r = 1$

(then we could consider the sign of $\frac{d^2r}{d\theta^2}$ to verify that there is a minimum at $r = 1$)

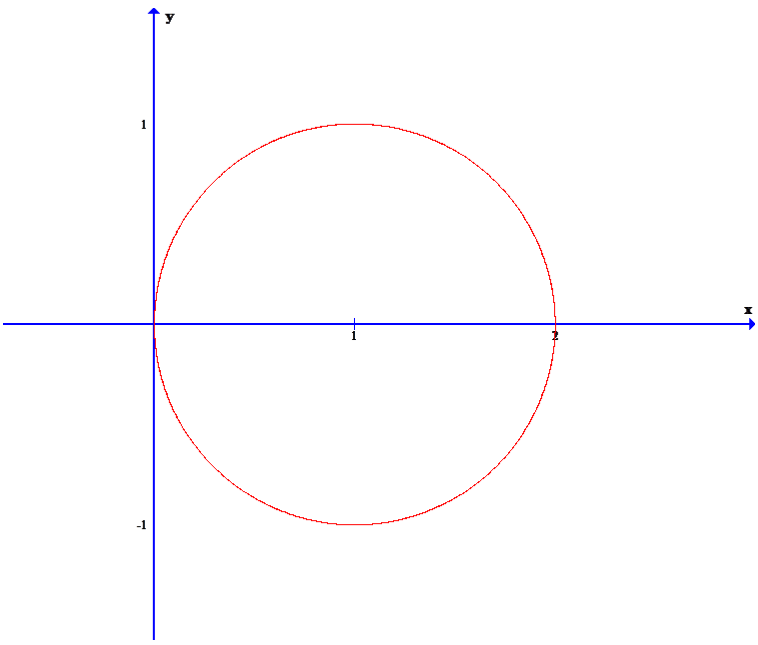
(14) **Example**: Find the Cartesian equation of the curve $r = 2cos\theta$

$x = rcos\theta\ ,\ y = rsin\theta$

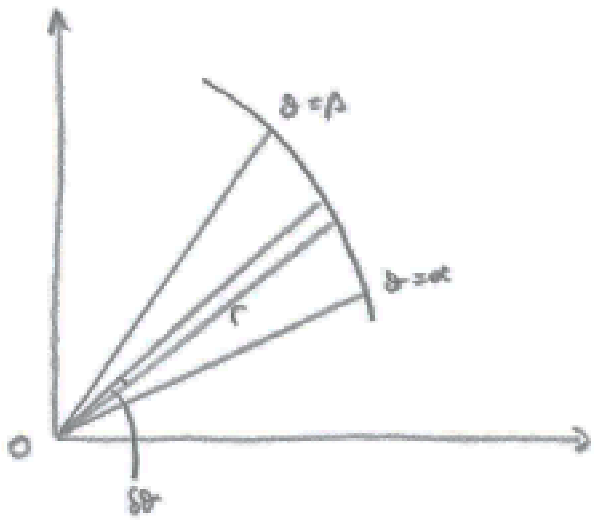
$\Rightarrow x = 2cos^2\theta,\ \ y = 2cos\theta sin\theta$

$\Rightarrow y^2 = 4cos^2\theta(1 - cos^2\theta) = x(2 - x) = 2x - x^2$

$\Rightarrow\ x^2 + y^2 - 2x = 0 \Rightarrow (x - 1)^2 + y^2 = 1$



(15) Area enclosed by a curve

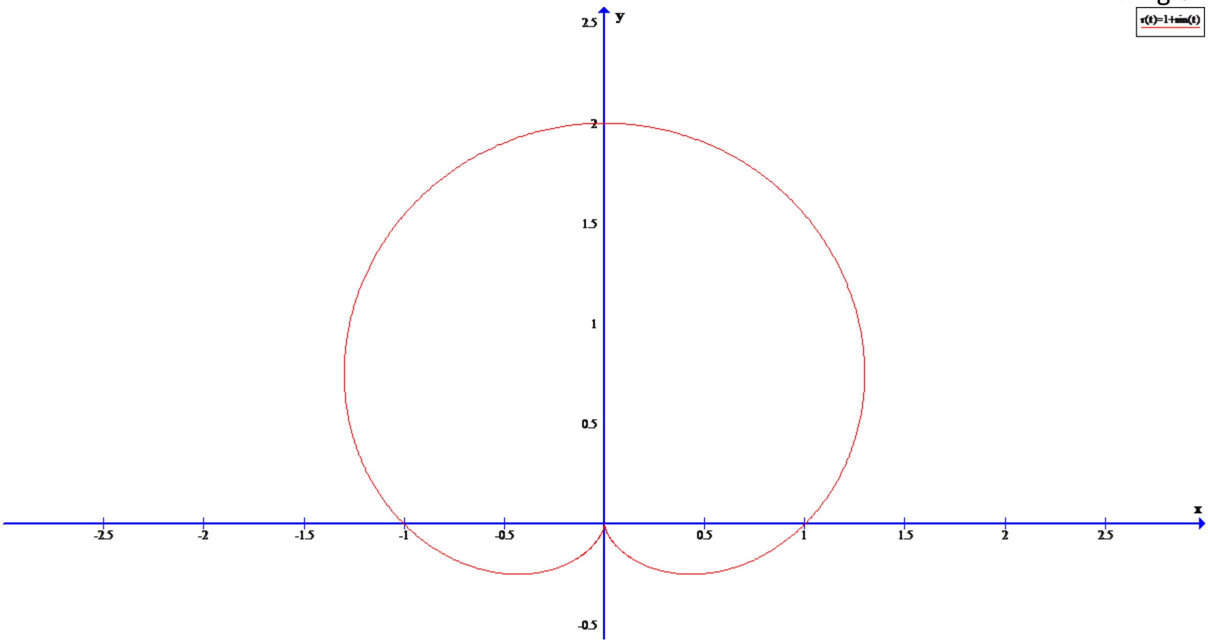


Referring to the diagram, the area of infinitesimal sector $= \frac{1}{2}(\delta\theta)r^2$

Hence required area $= \lim_{\delta\theta \to 0} \sum_{\alpha}^{\beta} \frac{1}{2}(\delta\theta)r^2 = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$

(16) **Example**: Find the area enclosed by

$r = 1 + sin\theta$ above the x axis



Area $= \int_0^{\pi} \frac{1}{2}(1 + sin\theta)^2 d\theta = \frac{1}{2}\int_0^{\pi} 1 + 2sin\theta + sin^2\theta \, d\theta$

$= \frac{1}{2}\int_0^{\pi} 1 + 2sin\theta + \frac{1}{2}(1 - cos2\theta) \, d\theta$

$= \frac{1}{2} \left[\frac{3\theta}{2} - 2cos\theta - sin2\theta\right]_0^{\pi}$

$= \frac{1}{2}\{[\frac{3\pi}{2} + 2 - 0] - [0 - 2 - 0]\}$

$= \frac{3\pi}{4} + 2$

Note: Unlike ordinary integration to find the area under a curve, for polar curves we don't need to worry about areas under the x axis contributing negative amounts (and therefore having to be dealt with separately).

(17) **Tangents to Curves**

Consider $r = 2cos\theta$

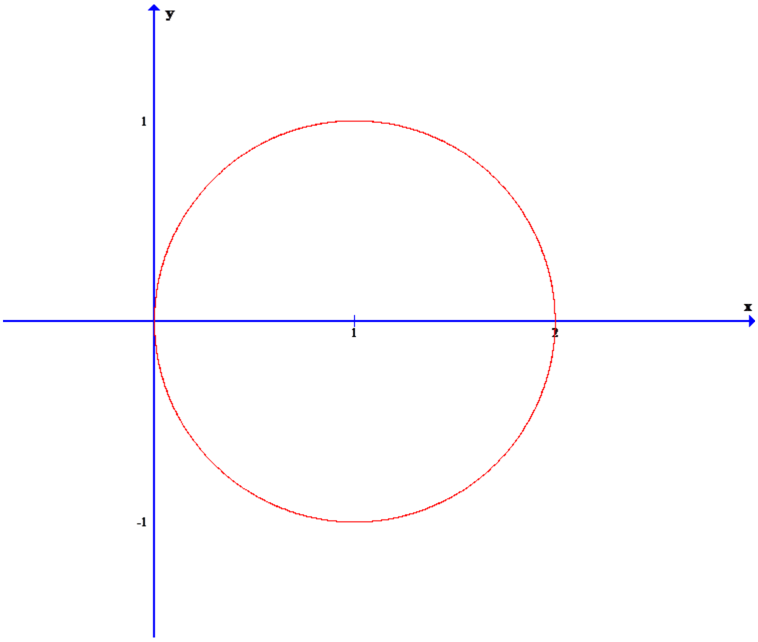
$x = rcos\theta$

$y = rsin\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

horizontal tangent $\Rightarrow \frac{dy}{d\theta} = 0$

vertical tangent $\Rightarrow \frac{dx}{d\theta} = 0$



$x = rcos\theta = 2cos^2\theta$

$y = rsin\theta = sin2\theta$

Exercise: Find where the tangents are horizontal and vertical.

Solution

$r = 2cos\theta$; $x = rcos\theta = 2cos^2\theta$; $y = rsin\theta = sin2\theta$

$\frac{dy}{d\theta} = 2cos2\theta$; horizontal tangent $\Rightarrow \frac{dy}{d\theta} = 0$

$\Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$\frac{dx}{d\theta} = 4cos\theta(-sin\theta) = -2sin2\theta$

vertical tangent $\Rightarrow \frac{dx}{d\theta} = 0$

$\Rightarrow 2\theta = 0, \pi, 2\pi, 3\pi$

$\Rightarrow \ \theta = 0, \frac{\pi}{2}\ , \pi, \frac{3\pi}{2}$

(18) To express equations of tangents in polar form

Horizontal tangent ($\theta = \frac{\pi}{4}$)

$y = a$ & $y = rsin\theta \Rightarrow rsin\theta = a \Rightarrow r = acosec\theta$

$r = 2cos\theta$; $\theta = \frac{\pi}{4} \Rightarrow r = 2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}$

So $\sqrt{2} = \frac{a}{\left(\frac{1}{\sqrt{2}}\right)} \Rightarrow a = 1$

Eq'n of tangent is $\mathrm{r} = \mathrm{cosec}\theta$

[Check: goes through (0,1)]

Exercise: Find eq'n of vertical tangent when $\theta = 0$

Solution

$x = a$ & $x = rcos\theta \Rightarrow rcos\theta = a \Rightarrow \mathrm{r} = asec\theta$

$r = 2cos\theta$; $\theta = 0 \Rightarrow r = 2(1) = 2$

So $2 = a(1) \Rightarrow a = 2$

Eq'n of tangent is $r = 2sec\theta$