Poisson Distribution – Exercises (with Sol'ns)

(12 pages; 17/2/26)

(1) Each week Isaac buys a lottery ticket, where the prize £X has a Poisson distribution with mean 1.

What is the probability that his average win over 10 weeks is more than £1?

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Solution

Let $Y = X_1 + X_2 + \cdots + X_{10}$, where each $X_i \sim Po(1)$.

Then required probability is

$P\left(\frac{Y}{10} > 1\right) = P(Y > 10)$

where $Y \sim Po(10)$

$= 1 - P(Y \leq 10)$,

$= 1 - 0.58304 = 0.41696$ or 0.417 (3sf)

(2) What are the (main) conditions for a Poisson model to be appropriate?

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Solution

(i) Events are random and occur independently of each other.

[This is normally treated as a single point, for exam purposes.]

(ii) Events occur at a uniform rate over the specified period.

[Not to be confused with the issue of λ being (unwisely) based on non-homogeneous data (involving different rates).]

(iii) If sample data are available, then the sample mean and variance should be close.

(3) If $X \sim Po(\lambda)$, what is the most common value of X?

(a) If $\lambda = 5$ (b) If $\lambda = 5.5$

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Solution

$$P(X = r) = e^{-\lambda}\frac{\lambda^r}{r!} = P(X = r - 1) \times \frac{\lambda}{r}$$

(a) Mode (most common value) occurs at $X = 4\ and\ X = 5$

(b) Mode occurs at $X = 5$

(4) The probability of there being a typographical error on a particular page of a 500 page book is assumed to be 0.09. Using a suitable approximation, what is the probability that at least one book out of 10 (each having 500 pages) contains more than 60 errors in total?

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Solution

Let X be the number of errors per book, so that $X \sim B(500, 0.09)$,

and as an approximation $X \sim Po(45)$.

Then $P(X > 60) = 1 - P(X \leq 60) = 1 - 0.98671 = 0.01329$

Let Y be the number of books with more than 60 errors,

so that $Y \sim B(10, 0.01329)$

Then $P(Y \geq 1) = 1 - P(Y = 0) = 1 - 0.98671^{10}$

$= 1 - 0.87477 = 0.12523 = 0.125$ (3sf)

(5) If a Binomial variable $X \sim B(n, p)$ is to be approximated by a Poisson variable, give an algebraic reason why p has to be small.

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Solution

For a Poisson variable, $Var = Mean$.

For the Binomial variable,

$Var = npq = np(1-p) \approx np = Mean$

(6) A random sample of 100 observations is taken from a Poisson distribution with mean 3.3

Find the (exact) probability that the mean of the sample is greater than 3.5

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Solution

$P\left(\frac{X_1+\cdots+X_{100}}{100} > 3.5\right) = P(X_1 + \cdots + X_{100} > 350)$

$= 1 - P(Y = X_1 + \cdots + X_{100} \leq 350)$

where $Y \sim Po(100 \times 3.3)$ ie $Po(330)$

And $1 - P(Y \leq 350) = 1 - 0.86996 = 0.13004$ or 0.130 (3sf)