Points of inflexion (3 pages; 24/10/18)

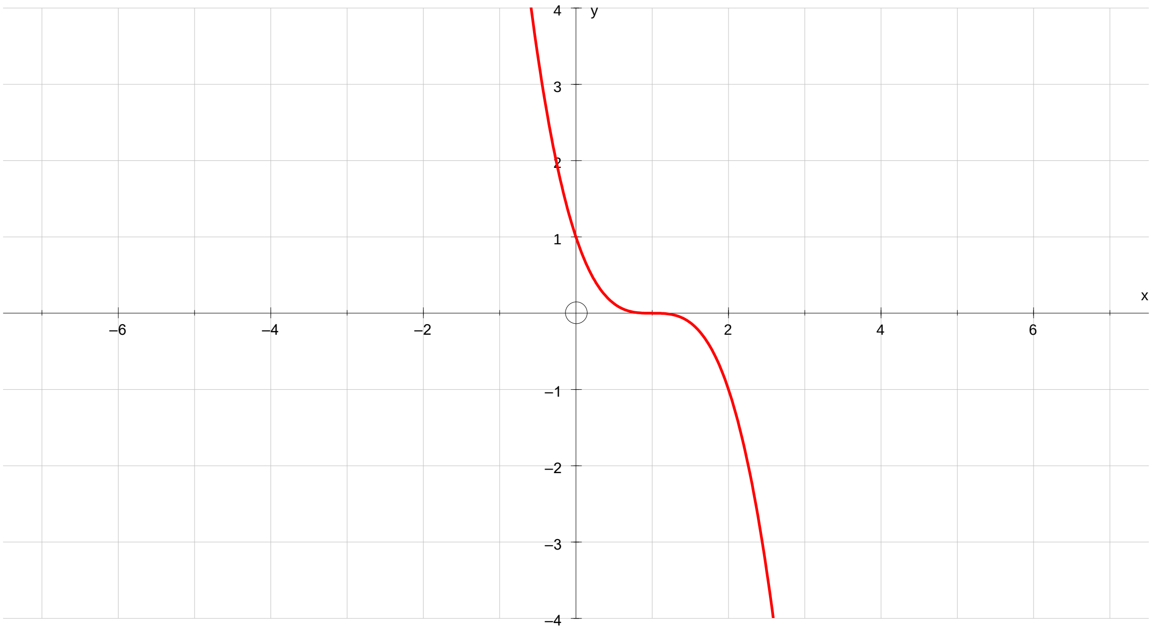
See also: "Turning points", "Cubic Functions"

(1) A point of inflexion occurs at the turning point of the gradient. A turning point occurs when the gradient changes sign (either from positive to negative, in the case of a maximum, or from negative to positive, in the case of a minimum). So a point of inflexion occurs when the gradient of the gradient changes sign; ie when $f''(x)$ changes sign. This is when a function changes from being convex to concave (or vice-versa). (See separate note "Convex & concave functions".)

(2) **Example 1** $y = (x-1)^3$

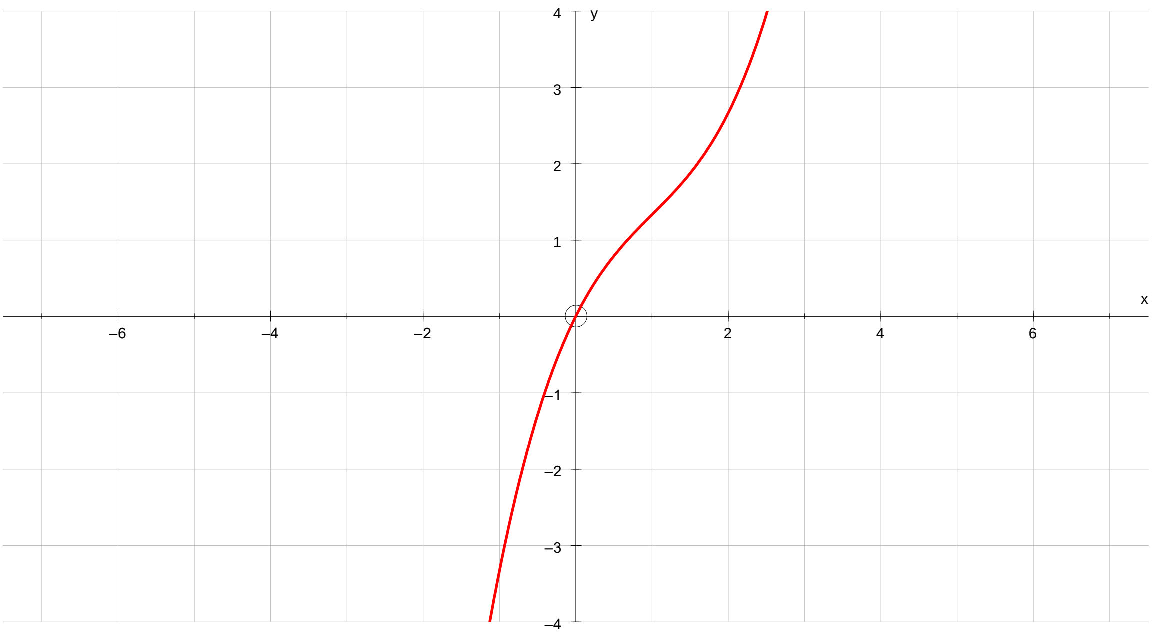
From left to right: $\frac{dy}{dx}$ is positive, falls to zero; then increases again; ie the gradient reaches a minimum.

Example 2: $y = (1 - x)^3$



From left to right: $\frac{dy}{dx}$ is negative, rises to zero; then decreases again; ie the gradient reaches a maximum.

Example 3: $y = \frac{1}{3}x^3 - x^2 + 2x$



From left to right: $\frac{dy}{dx}$ is positive, falls to 1 (at $x = 1$); then increases again; ie the gradient reaches a minimum.

Thus there is a point of inflexion at $x = 1$, which isn't a stationary point.

[This function was created as follows:

If $\frac{dy}{dx} = (x-1)^2 + 1$, then $\frac{dy}{dx}$ will have a minimum of 1 at $x = 1$;

y is then obtained by expanding and integrating $\frac{dy}{dx}$]

(3) (i) A necessary (but not sufficient) condition for a point of inflexion (turning point of the gradient) is that $\frac{d^2y}{dx^2} = 0$

Sufficient (but not necessary) conditions are $\frac{d^2y}{dx^2} = 0$ & $\frac{d^3y}{dx^3} \neq 0$

(4) Because a point of inflexion is a turning point of the gradient, a necessary and sufficient condition for a point of inflexion is that the first non-vanishing derivative must be odd (and of order at least 3).

Thus, in the case of $y = x^5 + x$ at $x = 0$,

$\frac{dy}{dx} = 1$, $\frac{d^2y}{dx^2} = 0$, $\frac{d^3y}{dx^3} = 0$, $\frac{d^4y}{dx^4} = 0$, $\frac{d^5y}{dx^5} = 120$

(5) A polynomial function of the form $(x-b)^{2n+1}h(x)$, where $n > 0$, has a point of inflexion at $(b, 0)$.