Angular Momentum & Planar Rigid Body Motion

(20 pages; 26/11/23)

Contents

(1) Types of rigid body motion

The most general type of rigid body motion concerns an irregular-shaped body, moving in 3D, and possessing both linear and rotational motion (both of which could involve acceleration). At any point in time, there will be an axis of rotation, and this could vary with time.

This note is restricted to the simpler case of 'planar' motion, concerning a 3D body that possesses reflective symmetry in a particular fixed plane, where all points of the body only move parallel to the fixed plane.

In general, the body can be considered to possess a combination of linear motion (possibly accelerating), and rotational motion about an axis through a certain point, perpendicular to the fixed plane (with possibly accelerating rotation).

The point is often the centre of mass of the body, but use can sometimes be made of the Instantaneous Centre of Rotation (discussed below).

A special case of planar motion is where the body is a lamina (ie has negligible width in the dimension perpendicular to the plane).

(2) Possible approaches

The simplest approach for dealing with a system of bodies (including particles) will often be an energy method. Otherwise the other main approach (excluding more advanced techniques) will be to set up equations of motion for each component of the system. These equations will be for either linear or rotational motion.

There may also be constraints on the motion (for example, it may be given that a cylinder is rolling, rather than slipping), and these can give rise to further equations.

As with equilibrium problems, it may be the case that not all of the available equations are needed; for example, if they involve forces that are not required to be found.

(3) Notation

The necessary theory is derived by treating the rigid body as though it were made up of individual particles.

The notation adopted in this note is as follows:

By default, vectors are relative to the Origin (O) of the frame of reference.

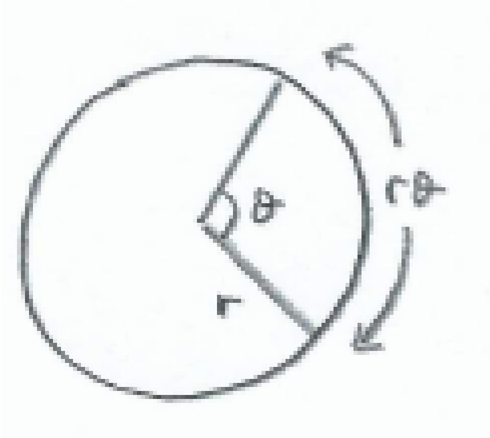
$\underline{r}_G$ is the position vector (relative to O) of the Centre of Mass (G) of a rigid body; $\underline{r}_i$ is the position vector (relative to O) of particle i; similarly for the velocity vectors $\underline{v}_G$ and $\underline{v}_i$.

$\underline{r}_{AB}$ is the position vector of B relative to A (and similarly for $\underline{v}_{AB}$); eg $\underline{r}_{GC}$ is the position vector of the Instantaneous Centre of Rotation (C) relative to the Centre of Mass, and $\underline{r}_{Gi}$ is the position vector of particle i relative to G.

(4) Angular Velocity

(1) Angular velocity of a particle

Consider a particle moving in a circular path, with (linear) speed $v(t)$.



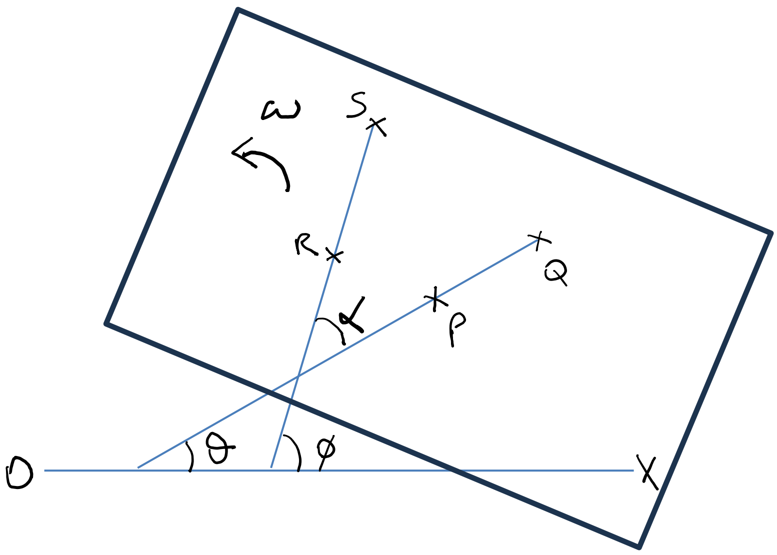
Referring to the diagram, arc length $s(t) = r\theta(t)$ [r is constant]

The angular speed is $\frac{d\theta(t)}{dt}$ or $\dot{\theta}$ (often denoted ω).

And $v(t) = \frac{ds(t)}{dt} = \frac{d}{dt}(r\theta(t)) = r\frac{d\theta(t)}{dt} = r\dot{\theta}$ or $r\omega$

The angular velocity is $\underline{v}(t) = r\omega\underline{k}$, where $\underline{k}$ is a unit vector (in the plane of the circle) in the direction of motion of the particle.

(2) Angular velocity of a lamina



Let OX be a line segment fixed in the frame of reference (see diagram). Let PQ be a line segment joining two points of the lamina, and RS a line segment joining two other points of the lamina. Let the angle between OX and PQ extended be θ, and the angle between OX and RS extended be ϕ, where $\phi = \theta + \alpha$. Note that α is defined by the intersection of the lines PQ and RS (both extended), and therefore α is fixed as the lamina turns.

Hence $\dot{\theta} = \dot{\phi}$, as α is fixed, and so each line of the lamina is turning at the same rate, and the angular velocity of the lamina can be defined uniquely as $\omega = \dot{\theta}$.

Thus the angular velocity is a measure of the lamina's rate of rotation relative to a fixed frame of reference (and this rate will in general vary with time). To establish ω, we can therefore consider the angle made by a line fixed in the lamina with a line fixed in the frame of reference (see θ in Example 12.2).

Also, there will usually be an Instantaneous Centre of Rotation, about which the lamina will (instantaneously) have angular velocity ω (discussed shortly).

(5) Moments of Inertia: see separate note

(6) Instantaneous Centre of Rotation

(6.1) Provided that there is some rotation of the body (in addition to any translational motion) [see note below], there will be at any time a point of the body (or an extension of it) that is instantaneously at rest relative to the plane of motion (although this point may have acceleration relative to the plane). This point is referred to as the Instantaneous Centre of Rotation (ICoR).

Note: If there is no rotation, then the ICoR can be thought of as being at infinity.

(6.2) As an example, for a hoop rolling on a surface, the ICoR is the point of contact with the surface. (See Example 12.1, which considers a cylinder.)

(6.3) The velocity of any point of a lamina is perpendicular to the line joining it to the ICoR.

The ICoR can sometimes be determined if there are two points on the body whose directions of motion are known (for example, the ends of a sliding ladder). The ICoR is then obtained from the intersection of these lines. See Example (12.2).

(7) Angular Momentum

(7.1) In its most general form, the angular momentum of a system can be determined about a particular point (for example, $\underline{L}_G$ if the point is the Centre of Mass, G). In the case of planar motion of a body with rotational symmetry about an axis through G that is perpendicular to the plane of motion (eg a rolling hoop), the angular momentum that will appear in an equation of motion is obtained by resolving $\underline{L}_G$ in the direction of the axis (to give $\underline{L}_G.\underline{n}$), as will be seen below (where $\underline{n}$ is a unit vector). Other components of $\underline{L}_G$ will then be zero, due to the symmetry.

(7.2) There are several points about which angular momentum could usefully be determined. $\underline{L}_O$ is the simplest form, but generally isn't that useful, except in deriving $\underline{L}_G$, which is the form that is most often used. $\underline{L}_C$ (where C is the ICoR) is also sometimes used.

(7.3) $\underline{L}_O = \sum_{i=1}^{N} m_i(\underline{r}_i \times \underline{v}_i)$ is the total angular momentum (or moment of momentum) of a system of N particles about (the fixed Origin) O. Note that $\underline{L}_O$ depends on the Origin chosen. However, a rigid body (or collection of particles) has an intrinsic angular momentum about G.

(7.4) $\underline{L}_G = \sum_{i=1}^{N} m_i([\underline{r}_i - \underline{r}_G] \times [\underline{v}_i - \underline{v}_G])$

$= \sum_{i=1}^{N} m_i([\underline{r}_i - \underline{r}_G] \times \underline{v}_i)$

$-\{\sum_{i=1}^{N} m_i([\underline{r}_i - \underline{r}_G]\} \times \underline{v}_G$

$= \sum_{i=1}^{N} m_i \underline{r}_i \times \underline{v}_i - \underline{r}_G \times \sum_{i=1}^{N} m_i \times \underline{v}_i$, as $\sum_{i=1}^{N} m_i[\underline{r}_i - \underline{r}_G] = \underline{0}$

$= \underline{L}_O - \underline{r}_G \times \underline{P}$, where $\underline{P} = \sum_{i=1}^{N} m_i \underline{v}_i = M\underline{v}_G$ is the total linear momentum of the system (where M is the total mass)

Thus $\underline{L}_G = \underline{L}_O - \underline{r}_G \times M\underline{v}_G$

(7.5) $\underline{L}_C = \sum_{i=1}^{N} m_i([\underline{r}_i - \underline{r}_C] \times [\underline{v}_i - \underline{v}_C])$

$= \sum_{i=1}^{N} m_i([\underline{r}_i - \underline{r}_C] \times \underline{v}_i)$, as $\underline{v}_C = \underline{0}$

$= \sum_{i=1}^{N} m_i \underline{r}_i \times \underline{v}_i - \underline{r}_C \times \sum_{i=1}^{N} m_i \times \underline{v}_i$

$= \underline{L}_O - \underline{r}_C \times \underline{P} = \underline{L}_O - \underline{r}_C \times M\underline{v}_G$

(7.6) $\underline{L}_C$ can also be determined from $\underline{L}_G$:

Result to prove: "The angular momentum of a body about the ICoR (C) equals the angular momentum about the Centre of Mass (G), plus the angular momentum of the total mass (assumed to be concentrated at G) about C."

In symbols, this is: $\underline{L}_C = \underline{L}_G + (\underline{r}_G - \underline{r}_C) \times M(\underline{v}_G - \underline{v}_C)$

Proof

RHS $= \underline{L}_G + (\underline{r}_G - \underline{r}_C) \times M\underline{v}_G$, as $\underline{v}_C = \underline{0}$

$= \left(\underline{L}_O - \underline{r}_G \times M\underline{v}_G\right) + \underline{r}_G \times M\underline{v}_G - \underline{r}_C \times M\underline{v}_G$

$= \underline{L}_O - \underline{r}_C \times M\underline{v}_G$

$= \underline{L}_C$, as required

(8) Torque

$\underline{\mathrm{K}}_O = \sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i$ is the total external torque about (the fixed Origin) O, where $\underline{F}_i$ is the external force acting on particle i

The total external torque about G, $\underline{\mathrm{K}}_G = \sum_{i=1}^{N} (\underline{r}_i - \underline{r}_G) \times \underline{F}_i$

$= \underline{\mathrm{K}}_0 - \underline{r}_G \times \underline{F}$, where $\underline{F} = \sum_{i=1}^{N} \underline{F}_i$

Similarly $\underline{\mathrm{K}}_C = \underline{\mathrm{K}}_0 - \underline{r}_C \times \underline{F}$

(9) Principle of Linear Momentum (Newton's 2nd Law):

$\underline{F} = M\ddot{\underline{r}}_G$, where $\underline{r}_G$ is the position vector of the centre of mass of a rigid body, M is the total mass of the body, and $\underline{F}$ is the net external force acting on the body

(10) Principle of Angular Momentum

(10.1) This can take two forms:

(i) The 1st form is $\underline{\mathrm{K}}_P = \frac{d\underline{\mathrm{L}}_P}{dt}$, where $\underline{\mathrm{K}}_P$ is the total torque about the point P, and $\underline{\mathrm{L}}_P$ is the angular momentum about P.

There are several possibilities for P:

(a) The (fixed) Origin, O (or in fact any fixed point): This is the simplest to derive, but is not usually of much practical use.

(b) The Centre of Mass, G: This is the form that is generally used.

(c) The Instantaneous Centre of Rotation, C: This is sometimes used.

(Note that C is instantaneously stationary relative to O, but not fixed for all times. The term 'fixed' implies stationary for all times.)

(ii) The 2nd form, which will generally be used when solving problems is

$\mathrm{K}_P = \frac{d}{dt}(I_P\omega) = I_P\dot{\omega}$, where $\mathrm{K}_P = |\underline{\mathrm{K}}_P|$

It is important to note that the axis about which the torque and angular momentum are calculated (giving rise to K_P and I_P, respectively) is not necessarily the same as the actual axis of rotation (though it will be parallel to it).

For example, if we are using $\mathrm{K}_G = \frac{d}{dt}(I_G\omega) = I_G\dot{\omega}$, then the actual axis of rotation will be passing through C, which need not be the same as G.

The main results for the Principle of Angular Momentum are:

$K_G = I_G\dot{\omega}$ and $K_C = I_C\dot{\omega}$

As a general rule, calculation of I_C is more complicated than for I_G (and typically may require calculation of I_G, and application of the Parallel Axis theorem). However, as will be seen in the Examples, use of I_C will generally enable N2L to be bypassed, so that less work is involved overall.

(10.2) Derivation of the 1st form

(10.2.1) For the Origin:

Applying Newton's 2nd Law to an individual particle i,

$\underline{F}_i = m_i\ddot{\underline{r}}_i$

(internal forces have been ignored, on the basis that they will cancel out when the whole system is considered)

Then $\frac{d\underline{\mathrm{L}}_0}{dt} = \frac{d}{dt}\left[\sum_{i=1}^{N} m_i\left(\underline{r}_i \times \underline{v}_i\right)\right]$

$= \sum_{i=1}^{N}\left[m_i\left(\dot{\underline{r}}_i \times \underline{v}_i\right) + m_i\left(\underline{r}_i \times \dot{\underline{v}}_i\right)\right]$

$= \sum_{i=1}^{N} m_i\left(\underline{r}_i \times \ddot{\underline{r}}_i\right)$, as $\dot{\underline{r}}_i \times \underline{v}_i = \underline{v}_i \times \underline{v}_i = \underline{0}$

$= \sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i = \underline{\mathrm{K}}_O$

Thus $\underline{\mathrm{K}}_O = \frac{d\underline{\mathrm{L}}_0}{dt}$, which is the Angular Momentum Principle applied to the Origin. However, it is not usually convenient to determine $\underline{\mathrm{K}}_O$ and $\underline{\mathrm{L}}_O$. Instead, corresponding results can be obtained for the Centre of Mass (G) and the ICoR (C):

(10.2.2) For the Centre of Mass

From (7.4), $\underline{L}_G = \underline{L}_O - \underline{r}_G \times M\underline{v}_G$ and from (8), $\underline{\mathrm{K}}_G = \underline{\mathrm{K}}_0 - \underline{r}_G \times \underline{F}$

Then $\underline{\mathrm{K}}_G = \frac{d\underline{\mathrm{L}}_0}{dt} - \underline{r}_G \times \underline{F} = \frac{d}{dt}(\underline{L}_G + \underline{r}_G \times M\underline{v}_G) - \underline{r}_G \times \underline{F}$

$= \frac{d}{dt}\underline{L}_G + \underline{\dot{r}}_G \times M\underline{v}_G + \underline{r}_G \times M\underline{\dot{v}}_G - \underline{r}_G \times \underline{F}$

$= \frac{d}{dt}\underline{L}_G + \underline{v}_G \times M\underline{v}_G + \underline{r}_G \times M\underline{\ddot{r}}_G - \underline{r}_G \times \underline{F}$

$= \frac{d}{dt}\underline{L}_G$, as $\underline{v}_G \times \underline{v}_G = \underline{0}$ and $\underline{F} = M\underline{\ddot{r}}_G$ (applying Newton's 2nd Law to G)

Thus $\underline{\mathrm{K}}_G = \frac{d\underline{\mathrm{L}}_G}{dt}$

(10.2.3) For the Instantaneous Centre of Rotation (ICoR):

From (7.5), $\underline{L}_C = \underline{L}_O - \underline{r}_C \times M\underline{v}_G$ and from (8), $\underline{\mathrm{K}}_C = \underline{\mathrm{K}}_0 - \underline{r}_C \times \underline{F}$

Then $\underline{\mathrm{K}}_C = \frac{d\underline{\mathrm{L}}_0}{dt} - \underline{r}_C \times \underline{F} = \frac{d}{dt}(\underline{L}_C + \underline{r}_C \times M\underline{v}_G) - \underline{r}_C \times \underline{F}$

$= \frac{d}{dt}\underline{L}_C + \underline{\dot{r}}_C \times M\underline{v}_G + \underline{r}_C \times M\underline{\dot{v}}_G - \underline{r}_C \times \underline{F}$

$= \frac{d}{dt}\underline{L}_C + \underline{r}_C \times (M\underline{\ddot{r}}_G - \underline{F})$, as $\underline{\dot{r}}_C = \underline{0}$

$= \frac{d}{dt}\underline{L}_C$, as $\underline{F} = M\underline{\ddot{r}}_G$

Thus $\underline{\mathrm{K}}_C = \frac{d}{dt}\underline{L}_C$

(10.3) Derivation of the 2nd form for the Centre of Mass

As mentioned earlier, for a rigid body with symmetry about the axis of rotation, the only non-zero component of $\underline{L}_G$ (for example) is the one in the direction of the axis: $\underline{L}_G.\underline{n}$ (where $\underline{n}$ is a unit vector).

The following diagrams are for a solid cylinder rolling on a horizontal surface (moving from left to right). The Instantaneous Axis of Rotation will pass through the contact of the cylinder with the surface.

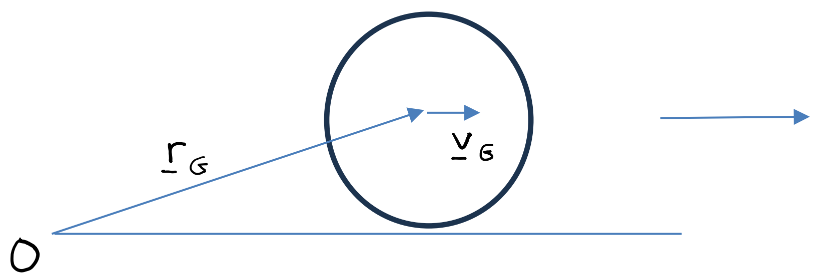


Diagram 1

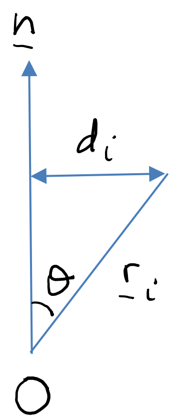


Diagram 2

From (7.4) again, $\underline{L}_G = \underline{L}_O - \underline{r}_G \times M\underline{v}_G$

and $\underline{L}_O = \sum_{i=1}^{N} m_i(\underline{r}_i \times \underline{v}_i)$

So $\underline{L}_G.\underline{n} = \{\sum_{i=1}^{N} m_i(\underline{r}_i \times \underline{v}_i).\underline{n}\} - (\underline{r}_G \times M\underline{v}_G).\underline{n}$

In Diagram 1, the axis of rotation $\underline{n}$ (which is directed into the paper) is perpendicular to both $\underline{r}_G$ and $\underline{v}_G$,

and so $(\underline{r}_G \times M\underline{v}_G).\underline{n} = 0$ (as $\underline{r}_G \times \underline{v}_G$ will be perpendicular to both $\underline{r}_G$ and $\underline{v}_G$).

Thus $\underline{L}_G.\underline{n} = \sum_{i=1}^{N} m_i(\underline{r}_i \times \underline{v}_i).\underline{n}$

$= \sum_{i=1}^{N} m_i\, \underline{n}\,.(\underline{r}_i \times \underline{v}_i)$

$= \sum_{i=1}^{N} m_i\underline{v}_i.(\underline{n} \times \underline{r}_i)$, by the cyclic interchange property of the

scalar triple product).

In Diagram 2 (from a different viewpoint), if the angular momentum is calculated with respect to an axis parallel to the axis of rotation (ie with direction $\underline{n}$) and passing through O:

$\underline{n} \times \underline{r}_i = |\underline{r}_i| sin\theta \underline{k}$, where $\underline{k}$ goes into the page (in the direction of travel of the particle) (as $\underline{n}$ is a unit vector),

$= d_i\underline{k}$

And $\underline{v}_i = |\underline{v}_i|\underline{k} = \omega d_i\underline{k}$, where ω is the angular speed of the body

Hence $\underline{L}_G.\underline{n} = \sum_{i=1}^{N} m_i\, \omega d_i\underline{k}.(d_i\underline{k})$

$= \sum_{i=1}^{N} m_i\, {d_i}^2\omega$

$= \omega \sum_{i=1}^{N} m_i\, {d_i}^2$

$= I_G\omega$, where I_G is the moment of inertia of the body about the centre of mass

So (as the only non-zero component of $\underline{L}_G$ is in the direction of $\underline{n}$, as discussed above), $\underline{L}_G = I_G\omega\underline{n}$ and hence $\underline{K}_G = \frac{d\underline{L}_G}{dt}$ becomes

$\underline{K}_G = |\underline{K}_G|\underline{n} = \frac{d}{dt}(I_G\omega\underline{n})$

so that $K_G = \frac{d}{dt}(I_G\omega) = I_G\dot{\omega}$ (where $K_G = |\underline{\mathrm{K}}_G|$), as $\underline{n}$ is fixed

(10.4) 2nd form for C

If instead the angular momentum is calculated with respect to the axis of rotation (passing through C, rather than G) then, in the same way, $\underline{L}_C.\underline{n} = I_C\omega$.

Then, with the torque also calculated with respect to the axis through C, $K_C = \frac{d}{dt}(I_C\omega) = I_C\dot{\omega}$

(11) Principle of Conservation of Energy

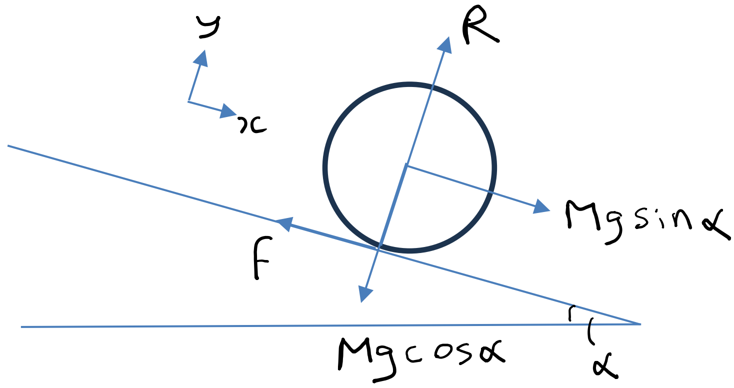
$T + V = constant$,

where $T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}I_G\omega^2$ and V is the gravitational potential energy

($\frac{1}{2}m\dot{r}^2$ is the kinetic energy of translation, and $\frac{1}{2}I_G\omega^2$ is the kinetic energy of rotation; with $\dot{r}$ being the speed of the centre of mass)

(12) Examples

(12.1) A solid cylinder of mass M and radius a rolls down an inclined plane. Show that the acceleration of the centre of mass of the cylinder down the slope is $\frac{2}{3}gsin\alpha$.



(See "Rolling Wheel – Friction" for a justification of the direction of the frictional force f.)

Approach 1 (Centre of Mass)

Applying N2L to the centre of mass (G): $Mgsin\alpha - f = M\ddot{x}_G$ (*)

Then, by the Angular Momentum principle, applied to G:

$K_G = \frac{d}{dt}(I_G\dot{\theta})$, where θ is the angle turned by the cylinder.

[The angular velocity $\dot{\theta}$ can be measured as the angular rate of turning of any line AB in the plane. Here A is G, and B is on the rim of the cylinder.]

And $K_G = af$ and $I_G = \frac{1}{2}Ma^2$,

so that $af = \frac{1}{2}Ma^2\ddot{\theta}$, or $f = \frac{1}{2}Ma\ddot{\theta}$

Also, as the cylinder is rolling, $x_G = a\theta$ (ie the distance covered on the sloping surface, and by G, is equal to the arc length corresponding to the angle θ).

Hence, $\ddot{x}_G = a\ddot{\theta}$, and so $f = \frac{1}{2}M\ddot{x}_G$

And substituting into (*) then gives

$Mgsin\alpha - \frac{1}{2}M\ddot{x}_G = M\ddot{x}_G$, so that $\frac{3}{2}\ddot{x}_G = gsin\alpha$, and $\ddot{x}_G = \frac{2}{3}gsin\alpha$, as required.

Approach 2 (Instantaneous Centre of Rotation)

The Instantaneous Centre of Rotation will be a point of contact of the cylinder with the surface.

The angular velocity $\dot{\theta}$ will be the same as in Approach 1.

Applying the Angular Momentum principle to C instead:

$K_C = I_C\ddot{\theta}$, with $K_C = aMgsin\alpha$

By the Parallel Axis theorem (see "Moments of Inertia"),

$I_C = I_G + Ma^2 = \frac{1}{2}Ma^2 + Ma^2 = \frac{3}{2}Ma^2$

and so $aMgsin\alpha = \frac{3}{2}Ma^2\ddot{\theta} = \frac{3}{2}Ma^2\frac{\ddot{x}_G}{a}$,

giving $\ddot{x}_G = \frac{2}{3}gsin\alpha$ again.

[Notice that we did not need to use N2L for this approach.]

Approach 3 (Conservation of Energy)

$\frac{1}{2}M{\dot{x}_G}^2 + \frac{1}{2}I_G\omega^2 - Mgxsin\alpha = Constant$ (1)

(G is a constant height above the point of contact of the cylinder with the surface, and $xsin\alpha$ measures the drop in height of the point of contact)

and $I_G = \frac{1}{2}Ma^2$;

also $x_G = a\theta$, so that $\omega = \dot{\theta} = \frac{\dot{x}_G}{a}$,

and hence, from (1),

$2\dot{x}_G{}^2 + a^2(\frac{\dot{x}_G}{a})^2 - 4gx_G sin\alpha = Constant$

or $3\dot{x}_G{}^2 - 4gx_G sin\alpha = Constant$

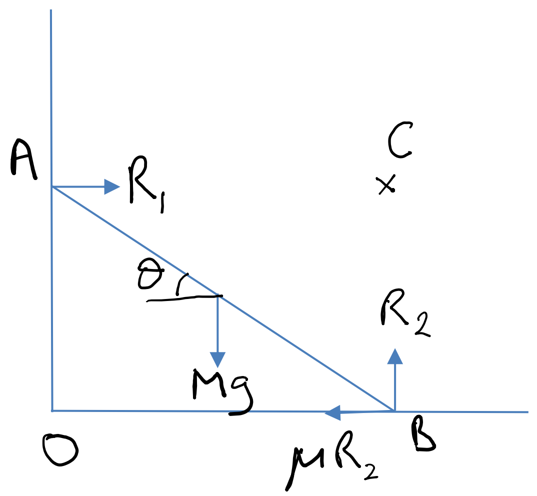
Then, differentiating wrt t,

$6\dot{x}_G\ddot{x}_G - 4g\dot{x}_G sin\alpha = 0$,

so that either $\dot{x}_G = 0$ (which can be rejected, as the cylinder is not stationary) or $\ddot{x}_G = \frac{2}{3}gsin\alpha$

[Once again, N2L is avoided.]

(12.2) A ladder sliding down a wall



where the ladder is of length $2a$, $x = OB$ and $y = OA$

In the case where both surfaces are smooth (so that $\mu = 0$), find a differential equation in θ, assuming that the ladder stays in contact with the two surfaces.

Approach 1 (Centre of Mass)

Applying N2L vertically to G: $R_2 - Mg = M\left(\frac{\ddot{y}}{2}\right)$ (1)

Applying N2L horizontally to G: $R_1 = M\left(\frac{\ddot{x}}{2}\right)$ (2)

Also, $x = 2acos\theta$ and $y = 2asin\theta$,

so that $\dot{x} = -2asin\theta.\dot{\theta}$ and $\dot{y} = 2acos\theta.\dot{\theta}$,

and hence $\ddot{x} = \left(-2acos\theta.\dot{\theta}\right)\dot{\theta} + (-2asin\theta)\ddot{\theta}$

and $\ddot{y} = \left(-2asin\theta.\dot{\theta}\right)\dot{\theta} + (2acos\theta)\ddot{\theta}$

Then from (1), $R_2 = Ma\left\{-sin\theta.\dot{\theta}^2 + cos\theta.\ddot{\theta}\right\} - Mg$, $(1')$

and from (2), $R_1 = Ma\{-cos\theta.\dot{\theta}^2 - sin\theta.\ddot{\theta}\}$ $(2')$

Applying the Angular Momentum principle to G:

$K_G = I_G\ddot{\theta}$, with $K_G = R_1asin\theta - R_2acos\theta$ (in direction of increasing θ)

and $I_G = \frac{1}{3}Ma^2$, so that $R_1asin\theta - R_2acos\theta \ = \frac{1}{3}Ma\ddot{\theta}$ (3)

Then, substituting into (3) for R_2 and R_1 from $(1')$ and $(2')$,

$Ma\left\{-cos\theta.\dot{\theta}^2 - sin\theta.\ddot{\theta}\right\}sin\theta$

$-\{Ma(-sin\theta.\dot{\theta}^2 + cos\theta.\ddot{\theta}) - Mg\}cos\theta = \frac{1}{3}Ma\ddot{\theta}$

so $\ddot{\theta}\left(-sin^2\theta - cos^2\theta - \frac{1}{3}\right) + \dot{\theta}^2(-cos\theta sin\theta + sin\theta cos\theta) = \frac{gcos\theta}{a}$

and hence $\ddot{\theta} = -\frac{3gcos\theta}{4a}$

[This can be solved by the standard method for a non-homogenous equation, with $\dot{\theta} = 0$ and $\theta = \alpha$ when $t = 0$.]

Approach 2 (Instantaneous Centre of Rotation)

As A is constrained to move vertically, C will lie on a line perpendicular to OA (so that A is instantaneously moving in a circle centred on C). Similarly, C will lie on a line perpendicular to OB.

Applying the Angular Momentum principle to C:

$K_C = I_C\ddot{\theta}$, with $K_C = -Mgacos\theta$ (in direction of increasing θ)

[As discussed earlier, the angular velocity $\dot{\theta}$ (and hence $\frac{d}{dt}\dot{\theta}$) is the same function of t, whether we are using $K_C = I_C\ddot{\theta}$ or

$K_G = I_G\ddot{\theta}$]

and $I_C = I_G + M(GC)^2$, by the Parallel Axis Theorem,

where $(GC)^2 = (\frac{x}{2})^2 + (\frac{y}{2})^2$

Hence $-Mgacos\theta = (\frac{1}{3}Ma^2 + \frac{M}{4}(x^2 + y^2))\ddot{\theta}$

$= (\frac{a^2}{3} + \frac{1}{4}(4a^2cos^2\theta + 4a^2sin^2\theta))M\ddot{\theta}$

$= \frac{4}{3}Ma^2\ddot{\theta}$

and so $\ddot{\theta} = \frac{-3gcos\theta}{4a}$ again

[Note how, once again, N2L did not need to be used. In this case, a lot of extra work has been avoided.]

Note: B and A are both performing instantaneous circular motion about C, and so $\omega\ or\ \dot{\theta} = \frac{-\dot{x}}{BC}$ (as $-\dot{x}$ is the speed of B in the direction of increasing θ) and also $\dot{\theta} = \frac{\dot{y}}{AC}$.

And hence $\dot{\theta} = \frac{-\dot{x}}{2asin\theta}$, so that $\dot{x} = -2asin\theta.\dot{\theta}$, which agrees with the working in Approach 1.

Also $\dot{\theta} = \frac{\dot{y}}{2acos\theta}$, so that $\dot{y} = 2acos\theta.\dot{\theta}$

Approach 3 (Conservation of Energy)

$\frac{1}{2}M\dot{r}_G{}^2 + \frac{1}{2}I_G\omega^2 + Mgy_G = Constant$ (1)

where $\dot{r}_G{}^2 = \dot{x}_G{}^2 + \dot{y}_G{}^2$,

and $x_G = \frac{x}{2} = acos\theta$ and $y_G = \frac{y}{2} = asin\theta$,

so that $\dot{x}_G = -asin\theta.\dot{\theta}$ and $\dot{y}_G = acos\theta.\dot{\theta}$,

and hence $\dot{r}_G{}^2 = a^2(\dot{\theta})^2$

And $I_G = \frac{1}{3}Ma^2$

Then, from (1),

$a^2(\dot{\theta})^2 + \frac{1}{3}a^2(\dot{\theta})^2 + g(2asin\theta) = Constant$,

or $\frac{4}{3}a(\dot{\theta})^2 + 2gsin\theta = Constant$

Then, differentiating wrt t,

$\frac{8}{3}a\dot{\theta}\ddot{\theta} + 2gcos\theta.\dot{\theta} = 0$,

so that either $\dot{\theta} = 0$ or $\ddot{\theta} = \frac{-3gcos\theta}{4a}$, as before,

with the solution $\dot{\theta} = 0$ being only valid at the start of the motion (as the ladder starts from rest).