Parabolas - Exercises (Solutions) (7 pages; 29/12/19)

(1***) Using the parametric equations of a parabola ($x = at^2$, $y = 2at$), show that the midpoints of chords of a parabola that have the same direction lie on a straight line parallel to the x-axis.

[A chord of a parabola joins two points on the parabola.]

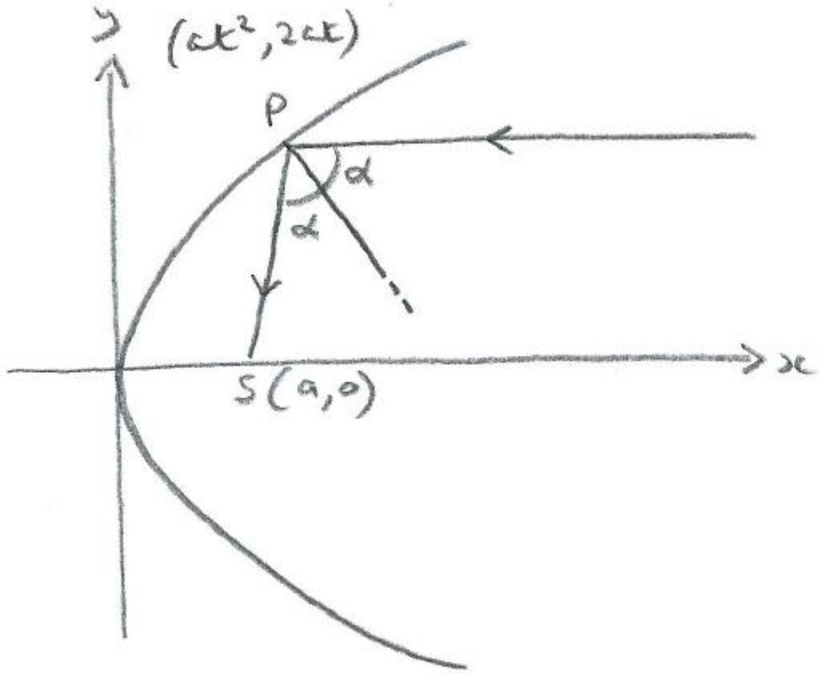
Solution

Let points P and Q on the parabola have parameters t_1 & t_2.

The chord PQ has gradient $\frac{2at_2-2at_1}{at_2{}^2-at_1{}^2} = \frac{2(t_2-t_1)}{t_2{}^2-t_1{}^2} = \frac{2}{t_1+t_2}$, and we are told that this is constant.

The y-coordinate of the midpoint of PQ is $\frac{1}{2}(2at_1 + 2at_2) = a(t_1 + t_2)$, which is constant, as $\frac{2}{t_1+t_2}$ and therefore $t_1 + t_2$ are constant, giving the required result.

(2***) A ray (eg of light) travels on a path parallel to the x-axis and hits the surface of the parabola $y^2 = 4ax$ at the point P $(at^2, 2at)$. The angle between the incoming ray and the normal at P is α. It can be assumed that the angle that the reflected ray makes with the normal is also α .



(i) Show that $tan\alpha = t$

(ii) Find the gradient of the reflected ray.

(iii) Show that the reflected ray passes through the focus of the parabola.

Solution

(i) The gradient of the tangent at P is $\frac{1}{t}$ (standard result), and hence the gradient of the normal is $-t$.

Hence, as the normal makes an angle $\pi - \alpha$ with the positive

x-axis, $-t = tan(\pi - \alpha) = -tan\alpha$, so that $tan\alpha = t$.

(ii) The gradient of the reflected ray is $\tan(\pi - 2\alpha) = -\tan(2\alpha)$

$= \frac{-2tan\alpha}{1-tan^2\alpha} = \frac{2t}{t^2-1}$

The equation of the reflected ray is $y - 2at = \frac{2t}{t^2-1}(x - at^2)$.

When it meets the x-axis, $-2at = \frac{2t}{t^2-1}(x - at^2)$, and $-a(t^2-1)=x-at^2$,

so that $x=a$; ie the reflected ray passes through the focus.

(3***) Suppose that $P\ (ap^2,2ap)$ and $Q\ (aq^2,2aq)$ are two points on the parabola $y^2=4ax$, such that the chord PQ passes through the focus of the parabola. Show that $pq=-1$.

Solution

If the focus is S, the gradient of PS equals the gradient of QS;

ie $\frac{2ap-0}{ap^2-a}=\frac{2aq-0}{aq^2-a}\Rightarrow\frac{p}{p^2-1}=\frac{q}{q^2-1}$

Treating this as a quadratic in p (for example),

$qp^2+p(1-q^2)-q=0$

$\Rightarrow\left(p+\frac{1}{q}\right)(qp-q^2)=0$

[as we trying to show that one root is $p=-\frac{1}{q}$]

$\Rightarrow p=-\frac{1}{q}\ or\ q(p-q)=0$

$q=0$ can be rejected, as PQ wouldn't pass through S, and $p\neq q$, otherwise PQ is not a chord

(4***) If the tangents to a parabola at P and Q are perpendicular, show that the chord PQ passes through the focus S of the parabola.

Solution

The gradients of the two tangents are $\frac{1}{p}$ and $\frac{1}{q}$ (standard result). As the tangents are perpendicular, $\left(\frac{1}{p}\right)\left(\frac{1}{q}\right)=-1$, so that $pq=-1$.

Gradient of PS $= \frac{2ap-0}{ap^2-a} = \frac{2p}{p^2-1}$, and the gradient of

QS $= \frac{2aq-0}{aq^2-a} = \frac{2q}{q^2-1}$

We wish to show that these gradients are the same; ie that

$\frac{2p}{p^2-1} = \frac{2q}{q^2-1}$

LHS $= \frac{2(-\frac{1}{q})}{(-\frac{1}{q})^2-1} = \frac{2q}{-1+q^2} = RHS$

(5***) Find the cartesian equations of the parabolas with:

(i) focus (4,4) and directrix $y = 0$

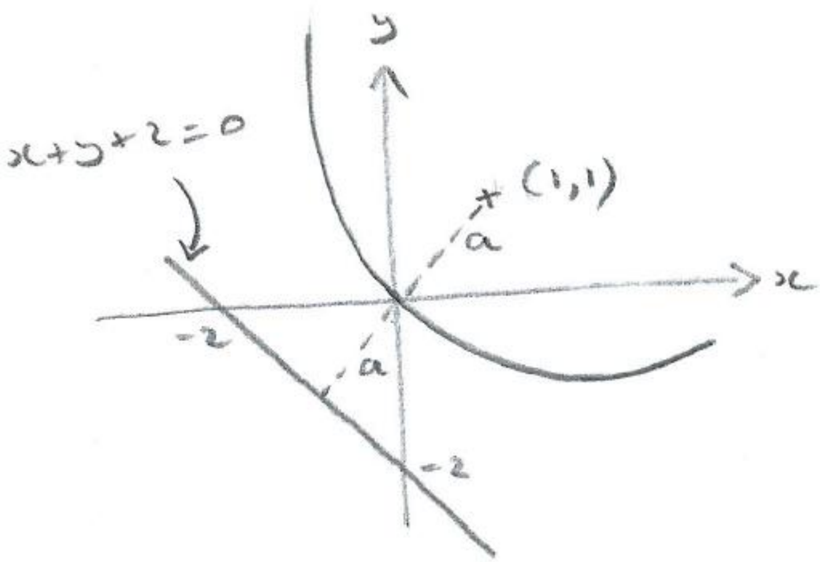
(ii) focus (1,1) and directrix $x + y + 2 = 0$

Solution

(i) The shortest distance from the focus to the directrix is $2a$, so $a = 2$.

Starting from the parabola with focus (0,2) and directrix $y = -2$ (with eq'n $x^2 = 4ay = 8y$), we need to make a translation of $\binom{4}{2}$, so that the eq'n becomes $(x-4)^2 = 8(y-2) = 8y - 16$

(ii) The directrix is $y = -x - 2$ (see diagram), and so $a = \sqrt{2}$



For general a, the parabola is obtained by rotating $y^2 = 4ax$ through 45° anti-clockwise.

Let the point (x, y) be transformed to the point (u, v) under the rotation.

Then $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} at^2 \\ 2at \end{pmatrix}$

$= \frac{1}{\sqrt{2}} \begin{pmatrix} at^2 - 2at \\ at^2 + 2at \end{pmatrix}$

Then $v - u = \frac{4at}{\sqrt{2}}$, so that $v = \frac{at}{\sqrt{2}}(t + 2) = \frac{a(\frac{\sqrt{2}(v-u)}{4a})}{\sqrt{2}} \left(\frac{\sqrt{2}(v-u)}{4a} + 2\right)$;

$4v(4a) = \sqrt{2}(v-u)^2 + 8a(v-u)$;

$\sqrt{2}(v-u)^2 = 8a(v+u)$;

$(v-u)^2 = 4a\sqrt{2}(u+v)$

In this case, $(v-u)^2 = 8(u+v)$,

which can be written as $(y-x)^2 = 8(x+y)$

[When $t = 1$ (at P, say), we expect PS to be parallel to the directrix (where S is the focus), by comparison with $y^2 = 4ax$.

When $t = 1$, $\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} at^2 - 2at \\ at^2 + 2at \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$,

and the gradient of PS is $\frac{3-1}{-1-1} = -1$, which is the gradient of the directrix.

(6***) Suppose that $P\ (ap^2, 2ap)$ and $Q\ (aq^2, 2aq)$ are two points on the parabola $y^2 = 4ax$, such that the chord PQ passes through the focus of the parabola. Show that the tangents at P and Q meet on the directrix (the result of Exercise (A)(3) can be assumed).

Solution

The tangents have equations $py - x = ap^2$ and $qy - x = aq^2$ [standard result], and $pq = -1$, from Exercise (A)(3).

The tangents meet the directrix when $x = -a$, so that for the tangent at P, $py + a = ap^2$ and hence $y = ap - \frac{a}{p}$, and for the tangent at Q, $y = aq - \frac{a}{q} = a\left(-\frac{1}{p}\right) + ap = ap - \frac{a}{p}$ also.

Thus the tangents meet on the directrix.

(7***) $P\ (ap^2, 2ap)$ and $Q\ (aq^2, 2aq)$ are two points on the parabola $y^2 = 4ax$, such that the chord PQ passes through the focus of the parabola. Show that the locus of the midpoint of PQ is a parabola, and establish its focus and directrix. (The result of Exercise (A)(3) can be assumed).

Solution

The midpoint of PQ is $\left(\frac{1}{2}a(p^2+q^2), \frac{1}{2}.2a(p+q)\right)$

Thus, for a point (x, y) on the locus of the midpoint,

$x = \frac{1}{2}a(p^2+q^2)$ and $y = a(p+q)$

Then $y^2 = a^2(p^2+q^2+2pq) = 2ax - 2a^2 = 4(\frac{a}{2})(x-a)$

This can be obtained from $y^2 = 4(\frac{a}{2})x$ by a translation of $\binom{a}{0}$, and so $y^2 = 4(\frac{a}{2})(x-a)$ is a parabola with vertex $(a, 0)$ and focus $\left(\frac{a}{2}+a, 0\right) = (\frac{3a}{2}, 0)$, and directrix $x = \frac{a}{2}$ [as the vertex is midway between the focus and the directrix]