Parabolas (10 pages; 16/7/19)

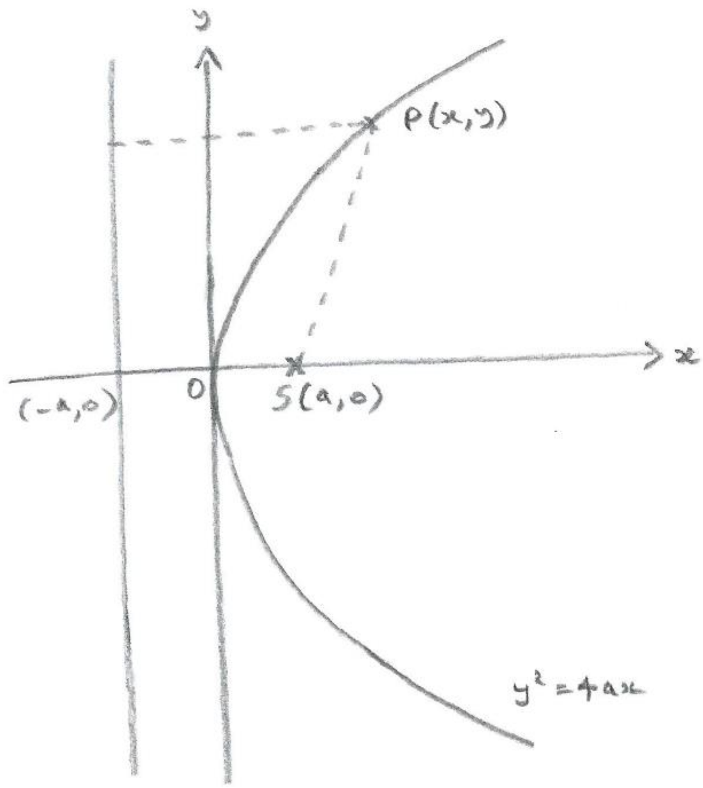
See "Conics" first, for features that are common to parabolas, ellipses and hyperbolas (as well as circles).

(1) Definitions

focus (S): (a,0) ($a > 0$)

directrix: $x = -a$

vertex: (0,0)



A parabola is the locus of points that are equidistant from the focus and the directrix.

The equation $y^2 = 4ax$ will be established in (2).

(2) Exercise

Given that the point (x, y) is equidistant from $(a, 0)$ and the line $x = -a$, show that $y^2 = 4ax$

Solution

Distance from $(a, 0)$ is $\sqrt{(x-a)^2 + (y-0)^2}$

Distance from the line $x = -a$ is $a + x$

So $\sqrt{(x-a)^2 + (y-0)^2} = a + x$

and hence $x^2 - 2ax + a^2 + y^2 = a^2 + 2ax + x^2$

so that $y^2 = 4ax$

(3) Parametric equations of a parabola

$x = at^2$, $y = 2at$ $(a > 0)$

So general point is $(at^2, 2at)$

$t = 0$ corresponds to the Origin;

$t > 0 \Rightarrow y > 0$ and $t < 0 \Rightarrow y < 0$

$t = \pm 1$ corresponds to $(a, \pm 2a)$

To confirm that the corresponding Cartesian equation is that of a parabola:

$y^2 = (2at)^2 = 4a^2t^2 = 4ax$

(4a) Equation of tangent

Show that the tangent at the point $(at^2, 2at)$ has equation

$ty - x = at^2$

Solution

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$

[Alternatively, $y^2 = 4ax \Rightarrow 2y.\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y} = \frac{2a}{\sqrt{4ax}} = \sqrt{\frac{a}{x}}$

$= \sqrt{\frac{a}{at^2}} = \frac{1}{t}$]

so that the eq'n of the tangent is $y - 2at = \frac{1}{t}(x - at^2)$

or $ty - 2at^2 = x - at^2$; ie $ty - x = at^2$

[Note that the gradient of the tangent tends to zero as $t \to \infty$.
Also that when $x = a$, the tangent is $y = x + a$ (when $t = 1$) or $y = -x - a$ (when $t = -1$).]

(4b) Equation of normal

Show that the normal at the point $(at^2, 2at)$ has equation
$y + tx = 2at + at^3$

Solution

Gradient of tangent is $\frac{1}{t}$, so gradient of normal is $-t$

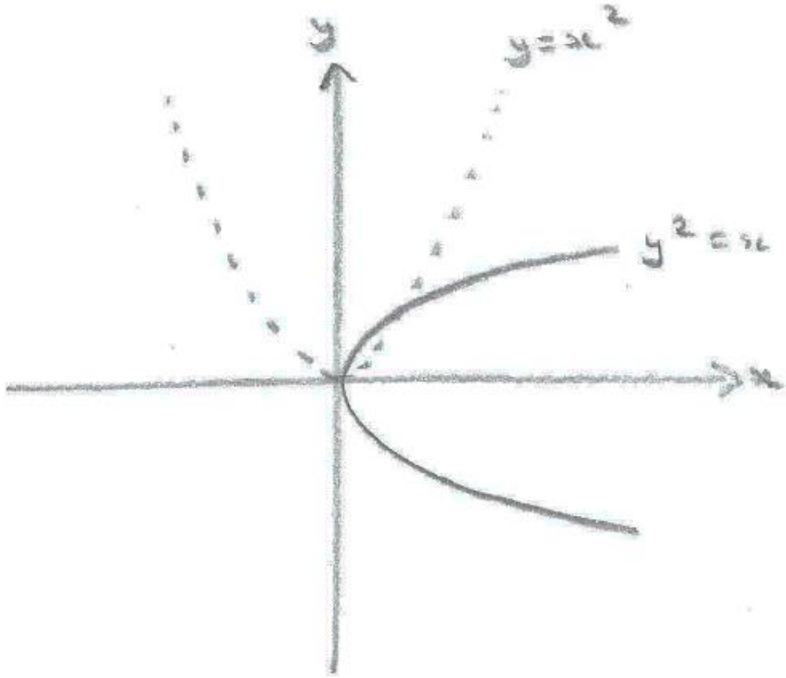
and the eq'n of the normal is $y - 2at = -t(x - at^2)$

or $y + tx = 2at + at^3$

[Note that the gradient of the normal tends to ∞ as $t \to \infty$.
Also that when $x = a$, the normal is $y = -x + 3a$ (when $t = 1$) or $y = x - 3a$ (when $t = -1$).]

(5) Transformations

Although it tends to be curves such as $y^2 = x$ (where $a = \frac{1}{4}$) that are usually cited as examples of parabolas, the inverse mapping $y = x^2$ is also a parabola.



Other parabolas can be obtained by translating, stretching or rotating $y^2 = x$.

For example, $(y-b)^2 = x-a$ results from a translation of $\binom{a}{b}$, whilst $(2b-y)^2 = 2a-x$ represents a rotation of 180° about the point (a,b). [See separate notes on transformations.]

Exercise Use matrices to show that the parabola $y^2 = 4ax$ becomes the parabola $x^2 = 4ay$ when it is rotated by 90° anti-clockwise.

Solution

Let the point (x,y) be transformed to the point (u,v) under the rotation.

Then $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} at^2 \\ 2at \end{pmatrix} = \begin{pmatrix} -2at \\ at^2 \end{pmatrix}$

To show that $u^2 = 4av$:

$u^2 = (-2at)^2 = 4a^2t^2$, and $4av = 4a(at^2) = 4a^2t^2$

Exercise Find the focus and directrix of the parabola

$y^2 = 12x - 24$

Solution

$y^2 = 12x - 24 = 4(3)(x - 2)$

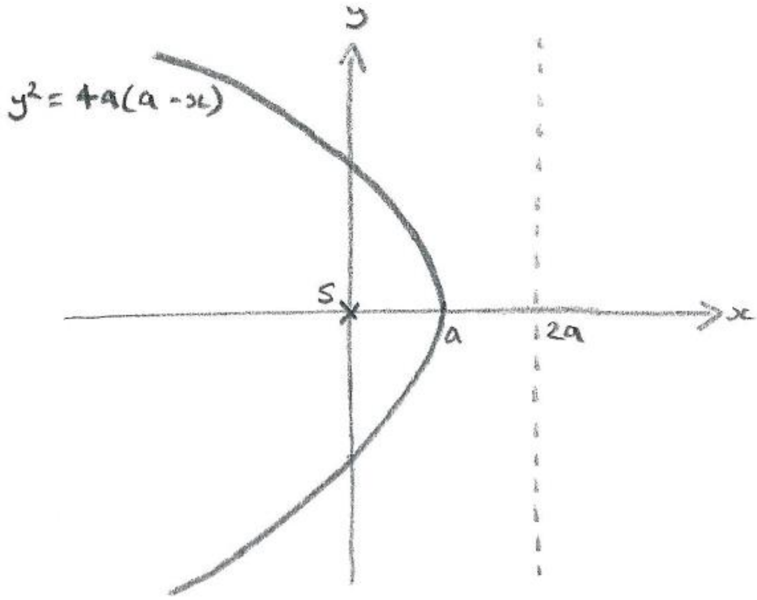
This can be obtained from $y^2 = 4(3)x$ by a translation of $\binom{2}{0}$, and so $y^2 = 12x - 24$ is a parabola with vertex (2,0) and focus

$(3 + 2,0) = (5,0)$, and directrix $x = -1$ [as the vertex is midway between the focus and the directrix]

(6) Polar form

See "Conics" for the derivation of the polar form of a general conic: $r = \frac{ep}{1+ecos\theta}$, where p is the (positive) distance between the focus and the directrix, for the case where the directrix is vertical and lies to the right of the pole.

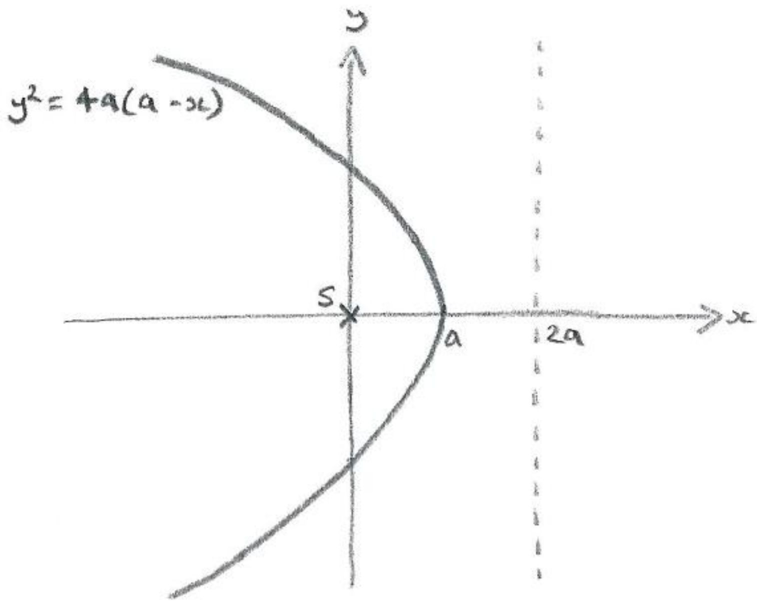
In order for the parabola to have the same location and orientation as the general conic used to derive the above formula, we reflect $y^2 = 4ax$ in the line $x = \frac{a}{2}$, to give $y^2 = 4a(a - x)$ [this can also be arrived at by translating by a to the left and then reflecting in the y-axis]. This means that the focus is moved to the Origin (see diagram below).



For this parabola, $e = 1$ and $p = 2a$, so that

$r = \frac{ep}{1+ecos\theta}$ becomes $r = \frac{2a}{1+cos\theta}$

(6a) Reconciliation with the Cartesian form of the parabola



As $x = rcos\theta$ and $y = rsin\theta$,

$y^2 = 4a(a - x) \Rightarrow r^2sin^2\theta = 4a^2 - 4arcos\theta$

$\Rightarrow r^2sin^2\theta + 4arcos\theta = 4a^2$

$\Rightarrow (rsin\theta + 2acot\theta)^2 - 4a^2cot^2\theta = 4a^2$

$\Rightarrow (rsin\theta + 2acot\theta)^2 = 4a^2cosec^2\theta$ (A)

$\Rightarrow rsin\theta + 2acot\theta = 2acosec\theta$

$\Rightarrow rsin^2\theta + 2acos\theta = 2a$

$\Rightarrow r = \frac{2a(1-cos\theta)}{(1-cos^2\theta)} = \frac{2a}{1+cos\theta}$

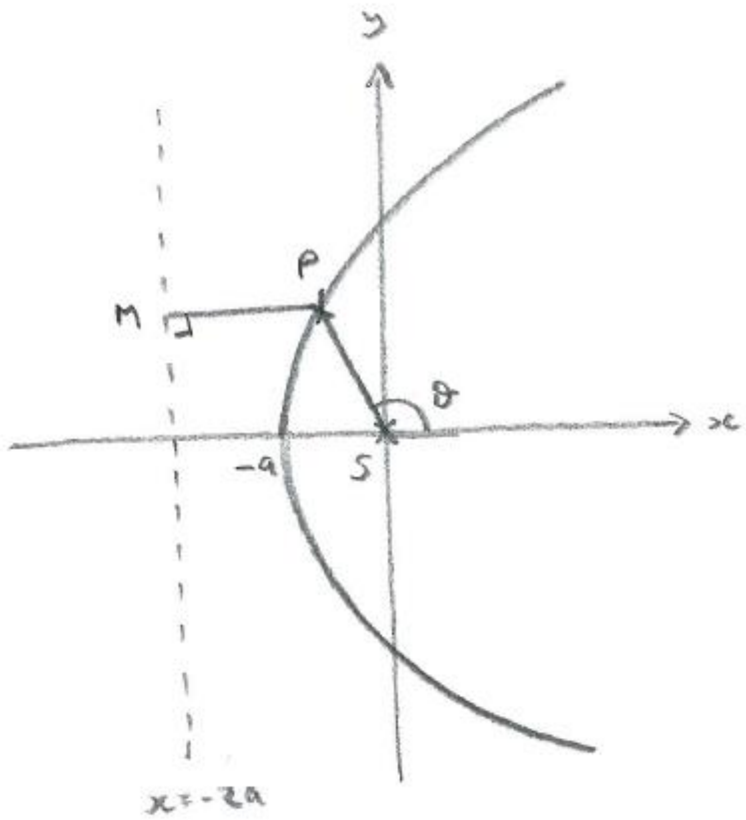
[Check: $\theta = 0 \Rightarrow r = a$ & $\theta = \frac{\pi}{2} \Rightarrow r = 2a$, which agrees with

$y^2 = 4a(a - x)$]

[The alternative solution of (A): $rsin\theta + 2acot\theta = -2acosec\theta$ is rejected, as it leads to $r = \frac{-2a}{1-cos\theta}$, which gives $r = -\infty$ when

$\theta = 0$]

Exercise: Find the appropriate polar form corresponding to the parabola shown below, and reconcile it with the Cartesian form.



The Cartesian form is $y^2 = 4a(x + a)$ (being $y^2 = 4ax$ translated a to the left).

Referring to the diagram above, $\frac{PS}{PM} = 1 \Rightarrow \frac{r}{2a - rcos(\pi - \theta)} = 1$

so that $2a + rcos\theta = r$ (as $\cos(\pi - \theta) = -cos\theta$)

and hence $r = \frac{2a}{1 - cos\theta}$

As $x = rcos\theta$ and $y = rsin\theta$,

$y^2 = 4a(x + a) \Rightarrow r^2 sin^2\theta = 4arcos\theta + 4a^2$

$\Rightarrow r^2 sin^2\theta - 4arcos\theta = 4a^2$

$\Rightarrow (rsin\theta - 2acot\theta)^2 - 4a^2 cot^2\theta = 4a^2$

$\Rightarrow (rsin\theta - 2acot\theta)^2 = 4a^2cosec^2\theta$ (A)

$\Rightarrow rsin\theta - 2acot\theta = 2acosec\theta$

$\Rightarrow rsin^2\theta - 2acos\theta = 2a$

$\Rightarrow r = \frac{2a(1+cos\theta)}{(1-cos^2\theta)} = \frac{2a}{1-cos\theta}$

[Check: $\theta = 0 \Rightarrow r = \infty$ & $\theta = \frac{\pi}{2} \Rightarrow r = 2a$, which agrees with

$y^2 = 4a(x + a)$]

(7) Summary of results [See Conics - Exercises]

(7.1) The gradient of the tangent at $(at^2, 2at)$ is $\frac{1}{t}$

(7.2) The midpoints of chords of a parabola that have the same direction lie on a straight line parallel to the x-axis.

(7.3) If a ray (eg of light) travels on a path parallel to the x-axis and hits the surface of the parabola $y^2 = 4ax$ at the point P $(at^2, 2at)$, then the reflected ray passes through the focus of the parabola.

(7.4a) If P $(ap^2, 2ap)$ and Q $(aq^2, 2aq)$ are two points on the parabola $y^2 = 4ax$, such that the chord PQ passes through the focus of the parabola [sometimes referred to as a **focal chord**], then $pq = -1$

(7.4b) If the tangents to a parabola at P and Q are perpendicular, then the chord PQ passes through the focus of the parabola.

(7.4c) If P $(ap^2, 2ap)$ and Q $(aq^2, 2aq)$ are two points on the parabola $y^2 = 4ax$, such that the chord PQ passes through the focus of the parabola, then the tangents at P and Q meet on the directrix.

(7.4d) If P $(ap^2, 2ap)$ and Q $(aq^2, 2aq)$ are two points on the parabola $y^2 = 4ax$, such that the chord PQ passes through the focus of the parabola, then the locus of the midpoint of PQ is the parabola $y^2 = 2a(x - a)$, with focus $(\frac{3a}{2}, 0)$ and directrix $x = \frac{a}{2}$