Probability Generating Function Q1 [Problem/H]
(12/6/21)

Given that $X_1, X_2, \dots, X_N$ & N are independent random variables, where the X_i are all distributed as X, and that

$S_N = X_1 + X_2 + \cdots + X_N$,

prove that $Var(S_N) = E(N)Var(X) + Var(N)[E(X)]^2$

The following results may be used:

(A) $E(X) = G'_X(1)$

(B) $VarX = G''_X(1) + G'_X(1) - [G'_X(1)]^2$

(C) $G_{S_N}(s) = G_N\big(G_X(s)\big)$

(D) $E(S_N) = E(N)E(X)$

Given that $X_1, X_2, \dots, X_N$ & N are independent random variables, where the X_i are all distributed as X, and that

$S_N = X_1 + X_2 + \cdots + X_N$,

prove that $Var(S_N) = E(N)Var(X) + Var(N)[E(X)]^2$

The following results may be used:

(A) $E(X) = G_X'(1)$

(B) $VarX = G_X''(1) + G_X'(1) - [G_X'(1)]^2$

(C) $G_{S_N}(s) = G_N\big(G_X(s)\big)$

(D) $E(S_N) = E(N)E(X)$

Solution

From (B), $Var(S_N) = G_{S_N}''(1) + G_{S_N}'(1) - \left[G_{S_N}'(1)\right]^2$

From (C), $G'_{S_N}(s) = G'_N\big(G_X(s)\big)G'_X(s)$

$= \{\frac{d}{du}G_N(u)\}\frac{du}{ds}$, where $u = G_X(s)$

Then $G''_{S_N}(s) = \frac{d}{ds}[\left\{\frac{d}{du}G_N(u)\right\}\frac{du}{ds}]$

$= \left\{\frac{d^2}{du^2}G_N(u)\right\}\frac{du}{ds}.\frac{du}{ds} + \left\{\frac{d}{du}G_N(u)\right\}\frac{d^2u}{ds^2}$

so that $G''_{S_N}(1) = G''_N(1)[G_X'(1)]^2 + G'_N\big(G_X(1)\big)G''_X(1)$

From (B), $VarN = G_N''(1) + G_N'(1) - [G_N'(1)]^2$,

so that $Var(S_N) = \{VarN - G_N'(1) + [G_N'(1)]^2\}[G_X'(1)]^2$

$+G'_N(1)G''_X(1) + G_{S_N}'(1) - \left[G_{S_N}'(1)\right]^2$, since $G_X(1) = 1$

$= \{VarN - E(N) + [E(N)]^2\}[E(X)]^2$

$+E(N)\{G''_X(1)\} + G_{S_N}'(1) - \left[G_{S_N}'(1)\right]^2$

Then, from (B), $VarX = G_X''(1) + G_X'(1) - [G_X'(1)]^2$

and, from (A), $E(X) = G_X'(1)$

Also, from (A)&(D), $G'_{S_N}(1) = E(N)E(X)$

Hence $Var(S_N) = \{VarN - E(N) + [E(N)]^2\}[E(X)]^2$

$+E(N)\{VarX - E(X) + [E(X)]^2\} + E(N)E(X) - [E(N)E(X)]^2$

$= VarN[E(X)]^2 + E(N)VarX$

$= E(N)Var(X) + Var(N)[E(X)]^2$, as required