PGFs - Exercises (11 pages; 26/2/25)

Given that $X_1, X_2, \dots, X_N$ & N are independent random variables, where the X_i are all distributed as X, and that

$S_N = X_1 + X_2 + \cdots + X_N$,

prove that $Var(S_N) = E(N)Var(X) + Var(N)[E(X)]^2$

The following results may be used:

(A) $E(X) = G_X'(1)$

(B) $VarX = G_X''(1) + G_X'(1) - [G_X'(1)]^2$

(C) $G_{S_N}(s) = G_N(G_X(s))$

(D) $E(S_N) = E(N)E(X)$

Solution

From (B), $Var(S_N) = G''_{S_N}(1) + G'_{S_N}(1) - \left[G'_{S_N}(1)\right]^2$

From (C), $G'_{S_N}(s) = G'_N\big(G_X(s)\big)G'_X(s)$

$= \{\frac{d}{du}G_N(u)\}\frac{du}{ds}$, where $u = G_X(s)$

Then $G''_{S_N}(s) = \frac{d}{ds}[\left\{\frac{d}{du}G_N(u)\right\}\frac{du}{ds}]$

$= \left\{\frac{d^2}{du^2}G_N(u)\right\}\frac{du}{ds}.\frac{du}{ds} + \left\{\frac{d}{du}G_N(u)\right\}\frac{d^2u}{ds^2}$

so that $G''_{S_N}(1) = G''_N(1)[G'_X(1)]^2 + G'_N\big(G_X(1)\big)G''_X(1)$

From (B), $VarN = G''_N(1) + G'_N(1) - [G'_N(1)]^2$,

so that $Var(S_N) = \{VarN - G'_N(1) + [G'_N(1)]^2\}[G'_X(1)]^2$

$+G'_N(1)G''_X(1) + G'_{S_N}(1) - \left[G'_{S_N}(1)\right]^2$, since $G_X(1) = 1$

$= \{VarN - E(N) + [E(N)]^2\}[E(X)]^2$

$+E(N)\{G''_X(1)\} + G'_{S_N}(1) - \left[G'_{S_N}(1)\right]^2$

Then, from (B), $VarX = G''_X(1) + G'_X(1) - [G'_X(1)]^2$

and, from (A), $E(X) = G'_X(1)$

Also, from (A)&(D), $G'_{S_N}(1) = E(N)E(X)$

Hence $Var(S_N) = \{VarN - E(N) + [E(N)]^2\}[E(X)]^2$

$+E(N)\{VarX - E(X) + [E(X)]^2\} + E(N)E(X) - [E(N)E(X)]^2$

$= VarN[E(X)]^2 + E(N)VarX$

$= E(N)Var(X) + Var(N)[E(X)]^2$, as required

A hen lays N eggs, where $N \sim P_o(\lambda)$, and each egg has probability p of hatching. Using any results about probability generating functions , show that the total number of eggs that hatch $\sim P_o(\lambda p)$

['Poisson hen']

Solution

Let the total number of eggs that hatch be $Z = X_1 + \cdots + X_N$,

where the $X_i \sim$Bernouilli(p).

Then $G_Z(s) = G_N\big(G_X(s)\big)$

with $G_N(s) = e^{\lambda(s-1)}$ and $G_X(s) = (1-p) + ps$,

so that $G_Z(s) = e^{\lambda(-p+ps)} = e^{\lambda p(s-1)}$

and hence $Z \sim P_o(\lambda p)$, as required.

Given that the pgf for the Binomial random variable $X \sim B(n, p)$ is $(q + ps)^n$, prove that $E(X) = np$ and $Var(X) = npq$

(where $q = 1 - p$) [standard results for pgfs can be used]

Solution

As $G'_X(s) = n(q+ps)^{n-1}p,\ E(X) = G'_X(1) = np$

And $G''_X(s) = n(n-1)(q+ps)^{n-2}p^2$,

so that $Var(X) = G''_X(1) + G'_X(1) - [G'_X(1)]^2$

$= n(n-1)p^2 + np - (np)^2$

$= np[(n-1)p + 1 - np]$

$= npq$

Establish the pgf for a Geometric random variable, and hence find $E(X)$ and $Var(X)$.

Solution

$G_X(s) = E(s^X) = \sum_{k=0}^{\infty} p_k s^k = \sum_{k=1}^{\infty} q^{k-1} p s^k$

$= ps \sum_{k=1}^{\infty} (qs)^{k-1} = \frac{ps}{1-qs}$ if $|qs| < 1$; ie $|s| < \frac{1}{q}$

$G'_X(s) = \frac{(1-qs)p - ps(-q)}{(1-qs)^2} = \frac{p}{(1-qs)^2}$

and $G''_X(s) = \frac{-2(-q)p}{(1-qs)^3} = \frac{2qp}{(1-qs)^3}$

Then $E(X) = G'_X(1) = \frac{1}{p}$

and $Var(X) = G''_X(1) + G'_X(1) - [G'_X(1)]^2$

$= \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2(1-p)+p-1}{p^2} = \frac{1-p}{p^2}\ or\ \frac{q}{p^2}$

Using the pgf for a Geometric random variable (and standard results for pgfs), find $E(Y)$ and $Var(Y)$ for a Negative Binomial random variable, $Y \sim nb(n, p)$.

Solution

$Y = X_1 + \cdots + X_n$, where $X_i \sim Geo(p)$

$G_{X_1+X_2}(s) = G_{X_1}(s)G_{X_2}(s)$, where X_1& X_2 are independent random variables

$G_{X_i}(s) = \frac{ps}{1-qs}$, so that (extending the above to a sum of n variables) $G_Y(s) = \left(\frac{ps}{1-qs}\right)^n$

Then $G_Y'(s) = n\left(\frac{ps}{1-qs}\right)^{n-1} . \frac{(1-qs)p-ps(-q)}{(1-qs)^2}$ (by the Quotient rule)

$= n\left(\frac{ps}{1-qs}\right)^{n-1} . \frac{p}{(1-qs)^2} = \frac{np^n s^{n-1}}{(1-qs)^{n+1}}$

and so $E(Y) = G_Y'(1) = \frac{np^n}{p^{n+1}} = \frac{n}{p}$

$Var(Y) = G_Y''(1) + G_Y'(1) - [G_Y'(1)]^2$

As $G_Y'(s) = \frac{np^n s^{n-1}}{(1-qs)^{n+1}}$,

$G_Y''(s) = \frac{np^n[(1-qs)^{n+1}(n-1)s^{n-2} - s^{n-1}(n+1)(1-qs)^n(-q)]}{(1-qs)^{2(n+1)}}$

(by the Quotient rule),

and hence $G_Y''(1) = \frac{np^n[p^{n+1}(n-1)-(n+1)p^n(-q)]}{p^{2(n+1)}}$

$= \frac{n[p(n-1)+(n+1)q]}{p^2}$

$= \frac{n[n(p+q)+q-p]}{p^2}$

$= \frac{n[n+q-p]}{p^2}$

Then $Var(Y) = G_Y''(1) + G_Y'(1) - [G_Y'(1)]^2$

$= \frac{n[n+q-p]}{p^2} + \frac{n}{p} - (\frac{n}{p})^2$

$= \frac{n[n+q-p+p-n]}{p^2}$

$= \frac{nq}{p^2}$