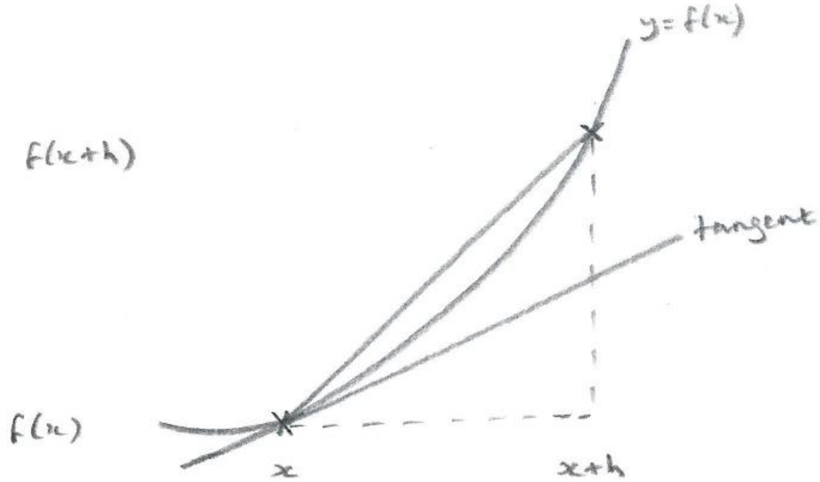
# Numerical Differentiation (8 pages; 31/3/20)

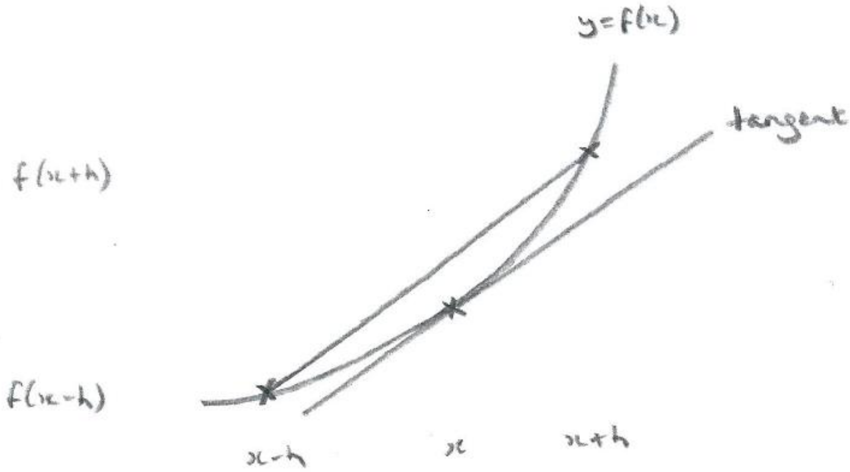
(1) Forward difference approximation to the derivative:

$f'(x) \approx \frac{f(x+h)-f(x)}{h}$ (for small $h$)




(2) Central difference approximation to the derivative:

$f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$



(3) Example: $f(x) = sinx$

To find $f'(1)$ using the forward difference and central difference methods:

| | A | B | C |
|---|---|---|---|
| 1 | | $\frac{\sin(1+h)-\sin(1)}{h}$ | $\frac{\sin(1+h)-\sin(1-h)}{2h}$ |
| 2 | h | | |
| 3 | | | |
| 4 | 0.1 | 0.497363753 | 0.539402252 |
| 5 | 0.01 | 0.536085981 | 0.540293301 |
| 6 | 0.001 | 0.53988148 | 0.540302216 |
| 7 | 0.0001 | 0.540260231 | 0.540302305 |
| 8 | 0.00001 | 0.540298099 | 0.540302306 |
| 9 | 0.000001 | 0.540301885 | 0.540302306 |
| 10 | 0.0000001 | 0.540302264 | 0.540302306 |
| 11 | 0.00000001 | 0.540302303 | 0.540302308 |

[Actual value: $f'(1) = \cos(1) = 0.540302306$]

(4) Exercise: Find an estimate for $f''(0)$ for $f(x) = sinx$, using $h = 0.01$, by (i) the Forward difference method, and (ii) the Central difference method.

**Solution**

(i) $f''(0) \approx \frac{f'(0.01)-f'(0)}{0.01}$

with $f'(0) \approx \frac{f(0.01)-f(0)}{0.01} = \frac{0.00999983-0}{0.01} = 0.999983$

and $f'(0.01) \approx \frac{f(0.02)-f(0.01)}{0.01} = \frac{0.0199987-0.00999983}{0.01} = 0.999887$,

so that $f''(0) \approx \frac{0.999887-0.999983}{0.01} = -0.0096$

[The actual value of $f''(0)$ is $-sin(0) = 0$]

(ii) $f''(0) \approx \frac{f'(0.01)-f'(-0.01)}{0.02}$,

with $f'(-0.01) \approx \frac{f(0)-f(-0.02)}{0.02} = \frac{0-(-0.0199987)}{0.02} = 0.999935$

and $f'(0.01) \approx \frac{f(0.02)-f(0)}{0.02} = \frac{0.0199987-0}{0.02} = 0.999935$,

so that $f''(0) \approx \frac{0.999935-0.999935}{0.02} = 0$

(5) The error for the forward difference can be shown to be approximately proportional to $h$; ie it is a "1st order" method. The error for the central difference method can be shown to be approximately proportional to $h^2$; ie it is a "2nd order" method (see Appendix). For this reason, the Central difference method is usually an improvement on the Forward difference method.

(6) A sequence is said to have 1st order convergence if

$e_{r+1} \approx ke_r$ (where $|k| < 1$), where $e_r = x_r - \alpha$

2nd order convergence is when $e_{r+1} \approx k{e_r}^2$ (again, with $|k| < 1$)

In the case of 1st order convergence, we can show that

$\frac{x_{r+1}-x_r}{x_r-x_{r-1}}$ (the 'ratio of differences') $\approx k$

**Proof**

$e_r = x_r - \alpha$ and $e_{r+1} = x_{r+1} - \alpha$

$e_r \approx ke_{r-1}$ and $e_{r+1} \approx ke_r$

So $\frac{x_{r+1}-x_r}{x_r-x_{r-1}} = \frac{(\alpha+e_{r+1})-(\alpha+e_r)}{(\alpha+e_r)-(\alpha+e_{r-1})} = \frac{e_{r+1}-e_r}{e_r-e_{r-1}} = \frac{ke_r-ke_{r-1}}{e_r-e_{r-1}} \approx k$

(7) Let $fd(h) = \frac{f(x+h)-f(x)}{h}$

As the forward difference method is a 1st order method,

$$\frac{fd\left(\frac{h}{2^n}\right)-f'(x)}{fd\left(\frac{h}{2^{n-1}}\right)-f'(x)} \approx \frac{\lambda(\frac{h}{2^n})}{\lambda(\frac{h}{2^{n-1}})} = \frac{1}{2}$$

Thus $e_n \approx \frac{1}{2}e_{n-1}$, where $e_n = x_n - \alpha$, with $x_n = fd\left(\frac{h}{2^n}\right)$

and $\alpha = f'(x)$,

and so the forward difference method has 1st order convergence.

Hence, from the 'ratio of differences' result, $\frac{fd\left(\frac{h}{4}\right)-fd(\frac{h}{2})}{fd\left(\frac{h}{2}\right)-fd(h)} \approx \frac{1}{2}$

$\Rightarrow fd\left(\frac{h}{4}\right) - fd\left(\frac{h}{2}\right) \approx \frac{1}{2}(fd\left(\frac{h}{2}\right) - fd(h))$ (1)

$\Rightarrow fd\left(\frac{h}{4}\right) \approx fd\left(\frac{h}{2}\right) + \frac{1}{2}(fd\left(\frac{h}{2}\right) - fd(h))$ (1')

ie given $fd(h)$ & $fd\left(\frac{h}{2}\right)$, we can estimate $fd\left(\frac{h}{4}\right)$ (which will be a better estimate for $f'(x)$)

Similarly, $fd\left(\frac{h}{8}\right) \approx fd\left(\frac{h}{4}\right) + \frac{1}{2}(fd\left(\frac{h}{4}\right) - fd(\frac{h}{2}))$ (1'')

So, from (1') & (1),

$fd\left(\frac{h}{8}\right) \approx \left[fd\left(\frac{h}{2}\right) + \frac{1}{2}\left(fd\left(\frac{h}{2}\right) - fd(h)\right)\right]$

$+\frac{1}{2} . \frac{1}{2}(fd\left(\frac{h}{2}\right) - fd(h))$ (2)

And $fd\left(\frac{h}{16}\right) \approx fd\left(\frac{h}{8}\right) + \frac{1}{2}(fd\left(\frac{h}{8}\right) - fd(\frac{h}{4}))$ (3)

so that, from (3) & (2)

$fd\left(\frac{h}{16}\right) \approx \left[fd\left(\frac{h}{2}\right) + \frac{1}{2}\left(fd\left(\frac{h}{2}\right) - fd(h)\right) + \frac{1}{2^2}(fd\left(\frac{h}{2}\right) - fd(h))\ \right]$

$+\frac{1}{2}.\frac{1}{2}(fd\left(\frac{h}{4}\right) - fd\left(\frac{h}{2}\right))$

[as $fd\left(\frac{h}{8}\right) - fd\left(\frac{h}{4}\right) \approx \frac{1}{2}(fd\left(\frac{h}{4}\right) - fd\left(\frac{h}{2}\right))$, from (1'')]

$\approx fd\left(\frac{h}{2}\right) + \frac{1}{2}\left(fd\left(\frac{h}{2}\right) - fd(h)\right) + \frac{1}{2^2}(fd\left(\frac{h}{2}\right) - fd(h))$

$+\frac{1}{2^3}(fd\left(\frac{h}{2}\right) - fd(h))$, from (1)

So, by repeating this process,

$f'(x) \approx fd\left(\frac{h}{2}\right) + \frac{1}{2}\left(fd\left(\frac{h}{2}\right) - fd(h)\right) + \frac{1}{2^2}(fd\left(\frac{h}{2}\right) - fd(h))$

$+\frac{1}{2^3}\left(fd\left(\frac{h}{2}\right) - fd(h)\right) + \cdots$

$\approx fd\left(\frac{h}{2}\right) + \frac{1}{2}\left(fd\left(\frac{h}{2}\right) - fd(h)\right)\frac{1}{1-\frac{1}{2}}$

$= fd\left(\frac{h}{2}\right) + \left(fd\left(\frac{h}{2}\right) - fd(h)\right)$

Naturally we would use the best pair of forward difference estimates available,

when $f'(x) \approx fd\left(\frac{h}{2^n}\right) + \left(fd\left(\frac{h}{2^n}\right) - fd\left(\frac{h}{2^{n-1}}\right)\right)$

or $2fd\left(\frac{h}{2^n}\right) - fd\left(\frac{h}{2^{n-1}}\right)$

Note: The same result can be obtained more simply as follows:

$\frac{fd\left(\frac{h}{2^n}\right) - f'(x)}{fd\left(\frac{h}{2^{n-1}}\right) - f'(x)} \approx \frac{1}{2} \Rightarrow 2fd\left(\frac{h}{2^n}\right) - 2f'(x) \approx fd\left(\frac{h}{2^{n-1}}\right) - f'(x)$

$\Rightarrow f'(x) \approx 2fd\left(\frac{h}{2^n}\right) - fd\left(\frac{h}{2^{n-1}}\right)$

(8) Let $cd(h) = \frac{f(x+h)-f(x-h)}{2h}$

As the central difference method is a 2nd order method,

$\frac{cd\left(\frac{h}{2^n}\right)-f'(x)}{cd\left(\frac{h}{2^{n-1}}\right)-f'(x)} \approx \frac{\lambda(\frac{h}{2^n})^2}{\lambda(\frac{h}{2^{n-1}})^2} = \frac{1}{4}$

and so the central difference method has 1st order convergence, as $e_n \approx \frac{1}{4}e_{n-1}$

[Note the confusing terminology: 2nd order method, but 1st order convergence.]

In the same way as for the forward difference method, this leads to

$f'(x) \approx cd\left(\frac{h}{2}\right) + \frac{1}{4}\left(cd\left(\frac{h}{2}\right) - cd(h)\right) + \frac{1}{4^2}\left(cd\left(\frac{h}{2}\right) - cd(h)\right)$

$+ \frac{1}{4^3}\left(cd\left(\frac{h}{2}\right) - cd(h)\right) + \cdots$

$\approx cd\left(\frac{h}{2}\right) + \frac{1}{4}\left(cd\left(\frac{h}{2}\right) - cd(h)\right)\frac{1}{1-\frac{1}{4}}$

$= cd\left(\frac{h}{2}\right) + \frac{1}{3}\left(cd\left(\frac{h}{2}\right) - cd(h)\right)$

and more generally:

$f'(x) \approx cd\left(\frac{h}{2^n}\right) + \frac{1}{3}\left(cd\left(\frac{h}{2^n}\right) - cd\left(\frac{h}{2^{n-1}}\right)\right)$

or $\frac{4}{3}cd\left(\frac{h}{2^n}\right) - \frac{1}{3}cd\left(\frac{h}{2^{n-1}}\right)$

## Appendix

(i) Proof that Forward Difference method is a 1st order method

One of the forms of the Taylor expansion of a function is

$$f(x+a)=f(a)+xf'(a)+\frac{x^2f''(a)}{2!}+\frac{h^3f'''(a)}{3!}+\cdots$$

(using the notation of "Taylor Series")

This can be rewritten as

$$f(h+x_0)=f(x_0)+hf'(x_0)+\frac{h^2f''(x_0)}{2!}+\frac{h^3f'''(x_0)}{3!}+\cdots$$

and it can be shown that $\frac{h^2f''(x_0)}{2!}+\frac{h^3f'''(x_0)}{3!}+\cdots=\frac{h^2f''(\eta)}{2!}$,

where $x_0<\eta<x_0+h$,

so that $f(x_0+h)=f(x_0)+hf'(x_0)+\frac{h^2f''(\eta)}{2!}$

and hence $\frac{f(x_0+h)-f(x_0)}{h}=f'(x_0)+\frac{hf''(\eta)}{2!}$

Thus the error term $\frac{f(x_0+h)-f(x_0)}{h}-f'(x_0)$ is approximately proportional to $h$ [as $f''(\eta)$ will tend to $f''(x)$]; ie is of order $h$ (or $O(h)$), and so the Forward Difference method is a 1st order method (because the error is of order $h^1$).

(ii) Proof that Central Difference method is a 2nd order method

It can also be shown that

$f(x_0+h)=f(x_0)+hf'(x_0)+\frac{h^2f''(x_0)}{2!}+\frac{h^3f'''(\eta_1)}{3!}$ , where

$x_0<\eta_1<x_0+h$

Also $f(x_0-h)=f(x_0)+(-h)f'(x_0)+\frac{(-h)^2f'''(x_0)}{2!}+\frac{(-h)^3f'''(\eta_2)}{3!}$

Then $f(x_0 + h) - f(x_0 - h) = 2hf'(x_0) + \frac{h^3[f'''(\eta_1)+f'''(\eta_2)]}{3!}$

and hence $\frac{f(x_0+h)-f(x_0-h)}{2h} = f'(x_0) + \frac{h^2[f'''(\eta_1)+f'''(\eta_2)]}{12}$

Thus the error term for the Central Difference method is approximately proportional to $h^2$ (ie is $O(h^2)$), and so it is a 2nd order method.