Normal Distribution (9 pages; 21/2/17)

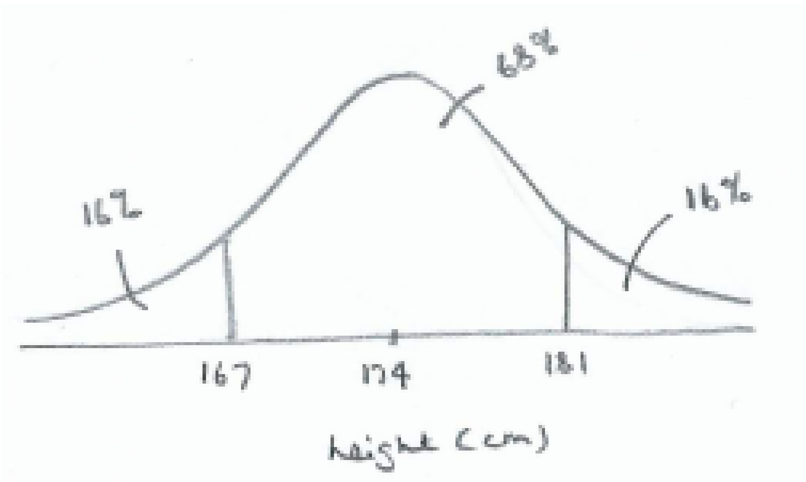
(1) The Normal distribution has a continuous, bell-shaped curve, often associated with naturally-occurring phenomena.

Example

Let X be the height (in cm) of an adult male in the UK.

Then, for the sake of argument, we will suppose that

$X \sim N(174, 49)$, so that $E(X) = 174$ and $Var(X) = 49$



(The approximate %s shown will be justified later on.)

(2) If $X \sim N(\mu, \sigma^2)$, then the probability density function (pdf) of X

is $\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$; $-\infty < x < \infty$

Its cumulative distribution function is given by:

$\Phi(a) = P(X < a) = \int_{-\infty}^{a} \phi(x) dx$

Notes

(i) There is no exact method for carrying out the integration for $\Phi(a)$, which has to determined approximately. (See "Maclaurin Expansions - Exercises")

(ii) $\int_{-\infty}^{\infty} \phi(x)dx = 1$

(iii) $\int_{-\infty}^{\mu} \phi(x)dx = 0.5$, as the Normal distribution is symmetric about the mean; ie $E(X)$

(iv) The height of the curve and the thickness of the tails will be determined by the variance.

(v) It can be shown that 1 standard deviation to either side of the mean corresponds to the point of inflexion of the curve (ie the turning point of the gradient). [See "Statistics Exercises"]

(3) It isn't feasible (or necessary) to have tables for combinations of μ & σ^2. Instead a transformation can be made to the 'standardised Normal' distribution, $Z \sim N(0,1)$, as follows:

Example

Suppose that the heights, X (in cm) of adult males in the UK are distributed $N(174, 49)$.

To find $P(X < 178)$:

$P(X < 178) = P\frac{X-174}{7} < \frac{178-174}{7}) = P(Z < 0.571)$

Thus, Z is obtained (from any Normal distribution) by first of all shifting the distribution, so that it is centred on a mean of 0, and then applying a scaling factor, so that the standard deviation becomes 1.

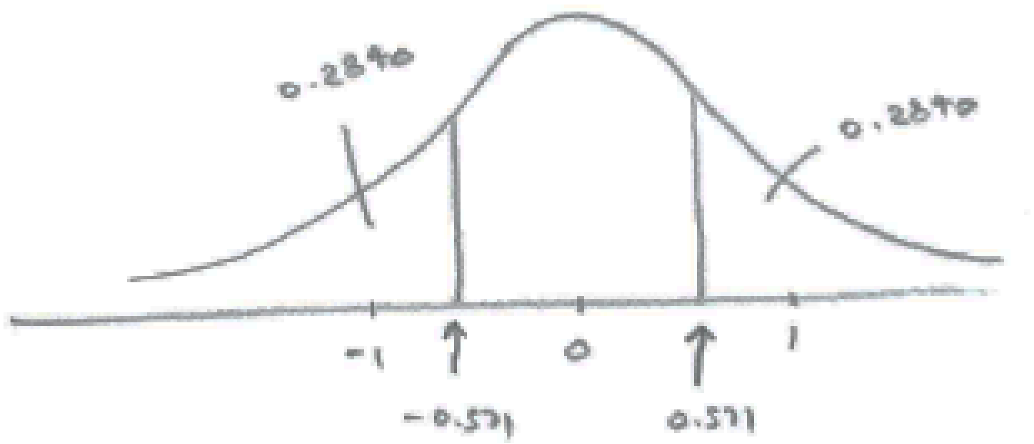
From the Normal tables (see Appendix),

$P(X < 178)) = P(Z < 0.571) = 0.7157 + 0.0003$

$= 0.7160 = 0.716$ (3sf)

Also, $P(X < 170) = P\frac{X-174}{7} < \frac{170-174}{7}) = P(Z < -0.571)$

$= 1 - P(Z < 0.571) = 1 - 0.7160 = 0.2840 = 0.284$ (3sf)



(4) Example (cont'd)

$P(165 < X < 185) = P(X < 185) - P(X < 165)$

$P(X < 185) = P\frac{X-174}{7} < \frac{185-174}{7}) = P(Z < 1.571)$

$= 0.9419$

$P(X < 165) = P\frac{X-174}{7} < \frac{165-174}{7}) = P(Z < -1.286)$

$= 1 - P(Z < 1.286)$

[by the symmetry of the curve, and the fact that the total area under the curve is 1]

$= 1 - 0.9008 = 0.0992$

Hence $P(165 < X < 185) = 0.9419 - 0.0992$

$= 0.8427 = 0.843$ (3sf)

(5) Inverse Normal Table

$\Phi(z) = P(Z < z) = p$ (say)

so that $z = \Phi^{-1}(p)$

eg $\Phi(1) = P(Z < 1) = 0.8413$ $and\ \Phi^{-1}(0.841) = 0.9986$

(unfortunately the table is limited to 3dp for p; hence the discrepancy between 0.9986 and 1)

See Appendix for the table for $\Phi^{-1}(p)$.

(6) Useful figures

$P(Z > 1) = 0.16$ (2sf)

$P(Z > 2) = 0.023$ (2sf)

$P(Z > 3) = 0.0013$ (2sf)

$P(Z > 1.645) = 0.05$

$P(Z > 1.96) = 0.025$

$P(Z > 2.326) = 0.01$

$P(Z > 2.576) = 0.005$

Note: As $P(Z > 1) = 0.16$; ie approximately 16% of the area under the standardised Normal curve lies to the right of 1, which is one standard deviation for $N(0,1)$, it follows that 16% of the area for any Normal distribution, $N(\mu,\sigma^2)$ lies to the right of one standard deviation; ie σ (as shown in (1)).

[A potential source of confusion here is the fact that 1 is the size of one standard deviation for $N(0,1)$. Thus, for example, $P(Z > 1.645) = 0.05$ could be written as

$P(Z > 0 + (1.645)(1)) = 0.05$

and $P(X > \mu + 1.645\sigma) = 0.05$, where $X \sim N(\mu,\sigma^2)$.]

(7) Example

If $P(X < 90) = 0.4$ and $P(X > 120) = 0.2$, find μ and σ, given that $X \sim N(\mu,\sigma^2)$

Solution

$P(X < 90) = P(\frac{X-\mu}{\sigma} < \frac{90-\mu}{\sigma})$

So $P(Z < \frac{90-\mu}{\sigma}) = 0.4$; ie left-hand tail of 40%

$\Rightarrow P\left(Z < -\left(\frac{90-\mu}{\sigma}\right)\right) = 0.6$

[where we expect $-\left(\frac{90-\mu}{\sigma}\right)$ to be positive]

$\Rightarrow \frac{-(90-\mu)}{\sigma} = \Phi^{-1}(0.6) = 0.2533 \quad (1)$

$P(X > 120) = P(\frac{X-\mu}{\sigma} > \frac{120-\mu}{\sigma})$

So $P(Z > \frac{120-\mu}{\sigma}) = 0.2$; ie right-hand tail of 20%

$\Rightarrow P(Z < \frac{120-\mu}{\sigma}) = 0.8$

$\Rightarrow \frac{120-\mu}{\sigma} = \Phi^{-1}(0.8) = 0.8416 \quad (2)$

and $\frac{-(90-\mu)}{\sigma} = 0.2533 \quad (1)$

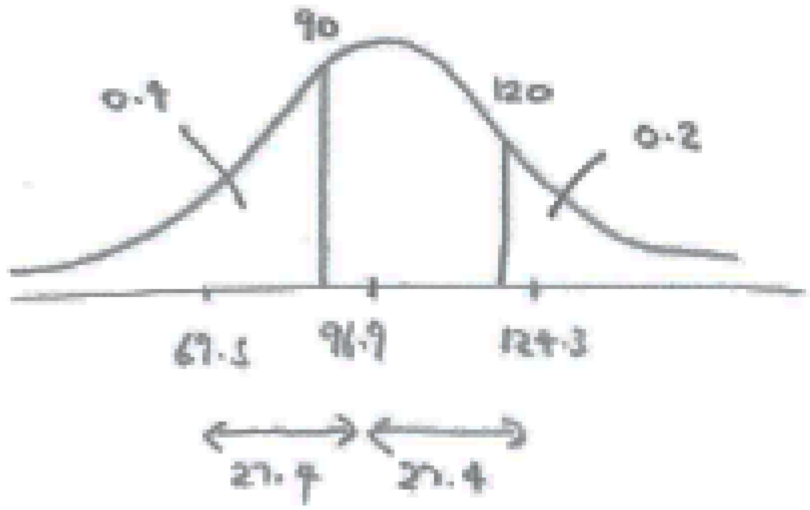
Solving (1) & (2) $\Rightarrow \frac{120-\mu}{\mu-90} = \frac{0.8416}{0.2533} = 3.3225$

$\Rightarrow 120 - \mu = 3.3225\mu - 299.025$

$\Rightarrow \mu = \frac{120+299.025}{3.3225+1} = 96.940 = 96.9\ (3sf)$

$(1) \Rightarrow \sigma = \frac{96.940-90}{0.2533} = 27.398 = 27.4\ (3sf)$

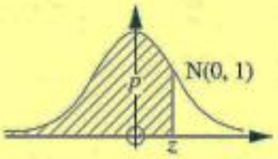
A reasonableness check can be made, by marking in one standard deviation either side of the mean (at the point of inflexion). Also, we know that roughly 16% of the area lies to the right of the 1 standard deviation point.



Appendix: Normal tables

The Normal distribution: values of $\Phi(z) = p$

The table gives the probability, p, of a random variable distributed as N(0, 1) *being less than z.*



(add)

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	1	2	3	4	5	6	7	8	9
0.0	.5000	5040	5080	5120	5160	5199	5239	5279	5319	5359	4	8	12	16	20	24	28	32	36
0.1	.5398	5438	5478	5517	5557	5596	5636	5675	5714	5753	4	8	12	16	20	24	28	32	35
0.2	.5793	5832	5871	5910	5948	5987	6026	6064	6103	6141	4	8	12	15	19	23	27	31	35
0.3	.6179	6217	6255	6293	6331	6368	6406	6443	6480	6517	4	8	11	15	19	23	26	30	34
0.4	.6554	6591	6628	6664	6700	6736	6772	6808	6844	6879	4	7	11	14	18	22	25	29	32
0.5	.6915	6950	6985	7019	7054	7088	7123	7157	7190	7224	3	7	10	14	17	21	24	27	31
0.6	.7257	7291	7324	7357	7389	7422	7454	7486	7517	7549	3	6	10	13	16	19	23	26	29
0.7	.7580	7611	7642	7673	7704	7734	7764	7794	7823	7852	3	6	9	12	15	18	21	24	27
0.8	.7881	7910	7939	7967	7995	8023	8051	8078	8106	8133	3	6	8	11	14	17	19	22	25
0.9	.8159	8186	8212	8238	8264	8289	8315	8340	8365	8389	3	5	8	10	13	15	18	20	23
1.0	.8413	8438	8461	8485	8508	8531	8554	8577	8599	8621	2	5	7	9	12	14	16	18	21
1.1	.8643	8665	8686	8708	8729	8749	8770	8790	8810	8830	2	4	6	8	10	12	14	16	19
1.2	.8849	8869	8888	8907	8925	8944	8962	8980	8997	9015	2	4	6	7	9	11	13	15	16
1.3	.9032	9049	9066	9082	9099	9115	9131	9147	9162	9177	2	3	5	6	8	10	11	13	14
1.4	.9192	9207	9222	9236	9251	9265	9279	9292	9306	9319	1	3	4	6	7	8	10	11	13
1.5	.9332	9345	9357	9370	9382	9394	9406	9418	9429	9441	1	2	4	5	6	7	8	10	11
1.6	.9452	9463	9474	9484	9495	9505	9515	9525	9535	9545	1	2	3	4	5	6	7	8	9
1.7	.9554	9564	9573	9582	9591	9599	9608	9616	9625	9633	1	2	3	3	4	5	6	7	8
1.8	.9641	9649	9656	9664	9671	9678	9686	9693	9699	9706	1	1	2	3	4	4	5	6	6
1.9	.9713	9719	9726	9732	9738	9744	9750	9756	9761	9767	1	1	2	2	3	4	4	5	5
2.0	.9772	9778	9783	9788	9793	9798	9803	9808	9812	9817	0	1	1	2	2	3	3	4	4
2.1	.9821	9826	9830	9834	9838	9842	9846	9850	9854	9857	0	1	1	2	2	2	3	3	4
2.2	.9861	9864	9868	9871	9875	9878	9881	9884	9887	9890	0	1	1	1	2	2	2	3	3
2.3	.9893	9896	9898	9901	9904	9906	9909	9911	9913	9916	0	1	1	1	1	2	2	2	2
2.4	.9918	9920	9922	9925	9927	9929	9931	9932	9934	9936	0	0	1	1	1	1	1	2	2
2.5	.9938	9940	9941	9943	9945	9946	9948	9949	9951	9952									
2.6	.9953	9955	9956	9957	9959	9960	9961	9962	9963	9964									
2.7	.9965	9966	9967	9968	9969	9970	9971	9972	9973	9974									
2.8	.9974	9975	9976	9977	9977	9978	9979	9979	9980	9981									
2.9	.9981	9982	9982	9983	9984	9984	9985	9985	9986	9986									
3.0	.9987	9987	9987	9988	9988	9989	9989	9989	9990	9990									
3.1	.9990	9991	9991	9991	9992	9992	9992	9992	9993	9993									
3.2	.9993	9993	9994	9994	9994	9994	9994	9995	9995	9995									
3.3	.9995	9995	9996	9996	9996	9996	9996	9996	9996	9997									
3.4	.9997	9997	9997	9997	9997	9997	9997	9997	9997	9998									

differences untrustworthy

The Inverse Normal function: values of $\Phi^{-1}(p) = z$

p	.000	.001	.002	.003	.004	.005	.006	.007	.008	.009
.50	.0000	.0025	.0050	.0075	.0100	.0125	.0150	.0175	.0201	.0226
.51	.0251	.0276	.0301	.0326	.0351	.0376	.0401	.0426	.0451	.0476
.52	.0502	.0527	.0552	.0577	.0602	.0627	.0652	.0677	.0702	.0728
.53	.0753	.0778	.0803	.0828	.0853	.0878	.0904	.0929	.0954	.0979
.54	.1004	.1030	.1055	.1080	.1105	.1130	.1156	.1181	.1206	.1231
.55	.1257	.1282	.1307	.1332	.1358	.1383	.1408	.1434	.1459	.1484
.56	.1510	.1535	.1560	.1586	.1611	.1637	.1662	.1687	.1713	.1738
.57	.1764	.1789	.1815	.1840	.1866	.1891	.1917	.1942	.1968	.1993
.58	.2019	.2045	.2070	.2096	.2121	.2147	.2173	.2198	.2224	.2250
.59	.2275	.2301	.2327	.2353	.2378	.2404	.2430	.2456	.2482	.2508
.60	.2533	.2559	.2585	.2611	.2637	.2663	.2689	.2715	.2741	.2767
.61	.2793	.2819	.2845	.2871	.2898	.2924	.2950	.2976	.3002	.3029
.62	.3055	.3081	.3107	.3134	.3160	.3186	.3213	.3239	.3266	.3292
.63	.3319	.3345	.3372	.3398	.3425	.3451	.3478	.3505	.3531	.3558
.64	.3585	.3611	.3638	.3665	.3692	.3719	.3745	.3772	.3799	.3826
.65	.3853	.3880	.3907	.3934	.3961	.3989	.4016	.4043	.4070	.4097
.66	.4125	.4152	.4179	.4207	.4234	.4261	.4289	.4316	.4344	.4372
.67	.4399	.4427	.4454	.4482	.4510	.4538	.4565	.4593	.4621	.4649
.68	.4677	.4705	.4733	.4761	.4789	.4817	.4845	.4874	.4902	.4930
.69	.4959	.4987	.5015	.5044	.5072	.5101	.5129	.5158	.5187	.5215
.70	.5244	.5273	.5302	.5330	.5359	.5388	.5417	.5446	.5476	.5505
.71	.5534	.5563	.5592	.5622	.5651	.5681	.5710	.5740	.5769	.5799
.72	.5828	.5858	.5888	.5918	.5948	.5978	.6008	.6038	.6068	.6098
.73	.6128	.6158	.6189	.6219	.6250	.6280	.6311	.6341	.6372	.6403
.74	.6433	.6464	.6495	.6526	.6557	.6588	.6620	.6651	.6682	.6713
.75	.6745	.6776	.6808	.6840	.6871	.6903	.6935	.6967	.6999	.7031
.76	.7063	.7095	.7128	.7160	.7192	.7225	.7257	.7290	.7323	.7356
.77	.7388	.7421	.7454	.7488	.7521	.7554	.7588	.7621	.7655	.7688
.78	.7722	.7756	.7790	.7824	.7858	.7892	.7926	.7961	.7995	.8030
.79	.8064	.8099	.8134	.8169	.8204	.8239	.8274	.8310	.8345	.8381
.80	.8416	.8452	.8488	.8524	.8560	.8596	.8633	.8669	.8705	.8742
.81	.8779	.8816	.8853	.8890	.8927	.8965	.9002	.9040	.9078	.9116
.82	.9154	.9192	.9230	.9269	.9307	.9346	.9385	.9424	.9463	.9502
.83	.9542	.9581	.9621	.9661	.9701	.9741	.9782	.9822	.9863	.9904
.84	.9945	.9986	1.003	1.007	1.011	1.015	1.019	1.024	1.028	1.032
.85	1.036	1.041	1.045	1.049	1.054	1.058	1.063	1.067	1.071	1.076
.86	1.080	1.085	1.089	1.094	1.099	1.103	1.108	1.112	1.117	1.122
.87	1.126	1.131	1.136	1.141	1.146	1.150	1.155	1.160	1.165	1.170
.88	1.175	1.180	1.185	1.190	1.195	1.200	1.206	1.211	1.216	1.221
.89	1.227	1.232	1.237	1.243	1.248	1.254	1.259	1.265	1.270	1.276
.90	1.282	1.287	1.293	1.299	1.305	1.311	1.317	1.323	1.329	1.335
.91	1.341	1.347	1.353	1.360	1.366	1.372	1.379	1.385	1.392	1.398
.92	1.405	1.412	1.419	1.426	1.433	1.440	1.447	1.454	1.461	1.468
.93	1.476	1.483	1.491	1.499	1.506	1.514	1.522	1.530	1.538	1.546
.94	1.555	1.563	1.572	1.581	1.589	1.598	1.607	1.616	1.626	1.635
.95	1.645	1.655	1.665	1.675	1.685	1.695	1.706	1.717	1.728	1.739
.96	1.751	1.762	1.774	1.787	1.799	1.812	1.825	1.838	1.852	1.866
.97	1.881	1.896	1.911	1.927	1.943	1.960	1.977	1.995	2.014	2.034
.98	2.054	2.075	2.097	2.120	2.144	2.170	2.197	2.226	2.257	2.290
.99	2.326	2.366	2.409	2.457	2.512	2.576	2.652	2.748	2.878	3.090

(see below for magnified version of LHS)

										(add)								
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	1	2	3	4	5	6	7	8	9
0.0	.5000	5040	5080	5120	5160	5199	5239	5279	5319	5359	4	8	12	16	20	24	28	32	36
0.1	.5398	5438	5478	5517	5557	5596	5636	5675	5714	5753	4	8	12	16	20	24	28	32	35
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0.8	.7881	7910	7939	7967	7995	8023	8051	8078	8106	8133	3	6	8	11	14	17	19	22	25
0.9	.8159	8186	8212	8238	8264	8289	8315	8340	8365	8389	3	5	8	10	13	15	18	20	23
1.0	.8413	8438	8461	8485	8508	8531	8554	8577	8599	8621	2	5	7	9	12	14	16	18	21
1.1	.8643	8665	8686	8708	8729	8749	8770	8790	8810	8830	2	4	6	8	10	12	14	16	19
1.2	.8849	8869	8888	8907	8925	8944	8962	8980	8997	9015	2	4	6	7	9	11	13	15	16
1.3	.9032	9049	9066	9082	9099	9115	9131	9147	9162	9177	2	3	5	6	8	10	11	13	14
1.4	.9192	9207	9222	9236	9251	9265	9279	9292	9306	9319	1	3	4	6	7	8	10	11	13
1.5	.9332	9345	9357	9370	9382	9394	9406	9418	9429	9441	1	2	4	5	6	7	8	10	11
1.6	.9452	9463	9474	9484	9495	9505	9515	9525	9535	9545	1	2	3	4	5	6	7	8	9
1.7	.9554	9564	9573	9582	9591	9599	9608	9616	9625	9633	1	2	3	3	4	5	6	7	8
1.8	.9641	9649	9656	9664	9671	9678	9686	9693	9699	9706	1	1	2	3	4	4	5	6	6
1.9	.9713	9719	9726	9732	9738	9744	9750	9756	9761	9767	1	1	2	2	3	4	4	5	5
2.0	.9772	9778	9783	9788	9793	9798	9803	9808	9812	9817	0	1	1	2	2	3	3	4	4
2.1	.9821	9826	9830	9834	9838	9842	9846	9850	9854	9857	0	1	1	2	2	2	3	3	4
2.2	.9861	9864	9868	9871	9875	9878	9881	9884	9887	9890	0	1	1	1	2	2	2	3	3
2.3	.9893	9896	9898	9901	9904	9906	9909	9911	9913	9916	0	1	1	1	1	2	2	2	2
2.4	.9918	9920	9922	9925	9927	9929	9931	9932	9934	9936	0	0	1	1	1	1	1	2	2

The Inverse Normal function: values of $\Phi^{-1}(p) = z$

p	.000	.001	.002	.003	.004	.005	.006	.007	.008	.009
.50	.0000	.0025	.0050	.0075	.0100	.0125	.0150	.0175	.0201	.0226
.51	.0251	.0276	.0301	.0326	.0351	.0376	.0401	.0426	.0451	.0476
.52	.0502	.0527	.0552	.0577	.0602	.0627	.0652	.0677	.0702	.0728
.53	.0753	.0778	.0803	.0828	.0853	.0878	.0904	.0929	.0954	.0979
.54	.1004	.1030	.1055	.1080	.1105	.1130	.1156	.1181	.1206	.1231
.55	.1257	.1282	.1307	.1332	.1358	.1383	.1408	.1434	.1459	.1484
.56	.1510	.1535	.1560	.1586	.1611	.1637	.1662	.1687	.1713	.1738
.57	.1764	.1789	.1815	.1840	.1866	.1891	.1917	.1942	.1968	.1993
.58	.2019	.2045	.2070	.2096	.2121	.2147	.2173	.2198	.2224	.2250
.59	.2275	.2301	.2327	.2353	.2378	.2404	.2430	.2456	.2482	.2508
.60	.2533	.2559	.2585	.2611	.2637	.2663	.2689	.2715	.2741	.2767
.61	.2793	.2819	.2845	.2871	.2898	.2924	.2950	.2976	.3002	.3029
.62	.3055	.3081	.3107	.3134	.3160	.3186	.3213	.3239	.3266	.3292
.63	.3319	.3345	.3372	.3398	.3425	.3451	.3478	.3505	.3531	.3558
.64	.3585	.3611	.3638	.3665	.3692	.3719	.3745	.3772	.3799	.3826
.65	.3853	.3880	.3907	.3934	.3961	.3989	.4016	.4043	.4070	.4097
.66	.4125	.4152	.4179	.4207	.4234	.4261	.4289	.4316	.4344	.4372
.67	.4399	.4427	.4454	.4482	.4510	.4538	.4565	.4593	.4621	.4649
.68	.4677	.4705	.4733	.4761	.4789	.4817	.4845	.4874	.4902	.4930
.69	.4959	.4987	.5015	.5044	.5072	.5101	.5129	.5158	.5187	.5215
.70	.5244	.5273	.5302	.5330	.5359	.5388	.5417	.5446	.5476	.5505
.71	.5534	.5563	.5592	.5622	.5651	.5681	.5710	.5740	.5769	.5799
.72	.5828	.5858	.5888	.5918	.5948	.5978	.6008	.6038	.6068	.6098
.73	.6128	.6158	.6189	.6219	.6250	.6280	.6311	.6341	.6372	.6403
.74	.6433	.6464	.6495	.6526	.6557	.6588	.6620	.6651	.6682	.6713
.75	.6745	.6776	.6808	.6840	.6871	.6903	.6935	.6967	.6999	.7031
.76	.7063	.7095	.7128	.7160	.7192	.7225	.7257	.7290	.7323	.7356
.77	.7388	.7421	.7454	.7488	.7521	.7554	.7588	.7621	.7655	.7688
.78	.7722	.7756	.7790	.7824	.7858	.7892	.7926	.7961	.7995	.8030
.79	.8064	.8099	.8134	.8169	.8204	.8239	.8274	.8310	.8345	.8381
.80	.8416	.8452	.8488	.8524	.8560	.8596	.8633	.8669	.8705	.8742
.81	.8779	.8816	.8853	.8890	.8927	.8965	.9002	.9040	.9078	.9116
.82	.9154	.9192	.9230	.9269	.9307	.9346	.9385	.9424	.9463	.9502
.83	.9542	.9581	.9621	.9661	.9701	.9741	.9782	.9822	.9863	.9904
.84	.9945	.9986	1.003	1.007	1.011	1.015	1.019	1.024	1.028	1.032
.85	1.036	1.041	1.045	1.049	1.054	1.058	1.063	1.067	1.071	1.076
.86	1.080	1.085	1.089	1.094	1.099	1.103	1.108	1.112	1.117	1.122
.87	1.126	1.131	1.136	1.141	1.146	1.150	1.155	1.160	1.165	1.170
.88	1.175	1.180	1.185	1.190	1.195	1.200	1.206	1.211	1.216	1.221
.89	1.227	1.232	1.237	1.243	1.248	1.254	1.259	1.265	1.270	1.276
.90	1.282	1.287	1.293	1.299	1.305	1.311	1.317	1.323	1.329	1.335
.91	1.341	1.347	1.353	1.360	1.366	1.372	1.379	1.385	1.392	1.398
.92	1.405	1.412	1.419	1.426	1.433	1.440	1.447	1.454	1.461	1.468
.93	1.476	1.483	1.491	1.499	1.506	1.514	1.522	1.530	1.538	1.546
.94	1.555	1.563	1.572	1.581	1.589	1.598	1.607	1.616	1.626	1.635
.95	1.645	1.655	1.665	1.675	1.685	1.695	1.706	1.717	1.728	1.739
.96	1.751	1.762	1.774	1.787	1.799	1.812	1.825	1.838	1.852	1.866
.97	1.881	1.896	1.911	1.927	1.943	1.960	1.977	1.995	2.014	2.034
.98	2.054	2.075	2.097	2.120	2.144	2.170	2.197	2.226	2.257	2.290
.99	2.326	2.366	2.409	2.457	2.512	2.576	2.652	2.748	2.878	3.090