Numerical Methods – Exercises (with Sol'ns)

(14 pages; 2/2/26)

Convergence

Complete the following table, where $e_r \approx k e_{r-1}$

$X_{4n} \approx X_{2n} + k(X_{2n} - X_n)$

$X_\infty \approx X_{2n} + \frac{1}{\frac{1}{k}-1}(X_{2n} - X_n)$

(where $X = T, M$ or S)

	Xth order method	Yth order convergence	k	$\frac{1}{\frac{1}{k}-1}$
Fixed Point	n/a		$g'(\alpha)$	n/a
Newton Raphson	n/a		n/a	n/a
Forward Difference				
Central Difference				
Trapezium rule				
Midpoint rule				
Simpson's rule				

Solution

	Xth order method	Yth order convergence	k	$\frac{1}{\frac{1}{k}-1}$
Fixed Point	n/a	1 (or 2 if $g'(\alpha)=0$)	$g'(\alpha)$	n/a
Newton Raphson	n/a	2	n/a	n/a
Forward Difference	1	1	$\frac{1}{2}$	1
Central Difference	2	1	$\frac{1}{4}$	$\frac{1}{3}$
Trapezium rule	2	1	$\frac{1}{4}$	$\frac{1}{3}$
Midpoint rule	2	1	$\frac{1}{4}$	$\frac{1}{3}$
Simpson's rule	4	1	$\frac{1}{16}$	$\frac{1}{15}$

Integration

Obtain the extrapolation formula $T \approx T_{2n} + \frac{1}{3}\,(T_{2n} - T_n)$

from $T_{2n} - T \approx \frac{T_n - T}{4}$

Obtain the extrapolation formula $T \approx T_{2n} + \frac{1}{3}\ (T_{2n} - T_n)$

from $T_{2n} - T \approx \frac{T_n - T}{4}$

Solution

$T_{2n} - T \approx \frac{T_n - T}{4}$

$\Rightarrow T_{2n} - \frac{T_n}{4} \approx \frac{3T}{4}$

$\Rightarrow T \approx \frac{4T_{2n}}{3} - \frac{T_n}{3} = T_{2n} + \frac{1}{3}\ (T_{2n} - T_n)$

$\int_0^{\frac{\pi}{2}} \sqrt{sinx}\ dx$ is to be estimated. Complete the following table, by the quickest method. Give values to 6 dp.

n	T_n	M_n	S_n
1			
2			
4			
8			

$\int_0^{\frac{\pi}{2}} \sqrt{sinx}\ dx$ is to be estimated. Complete the following table, by the quickest method. Give values to 6 dp.

Solution

Let $f(x) = \sqrt{sinx}$

$T_1 = \frac{1}{2}\left(\frac{\pi}{2}\right)\left(f(0) + f\left(\frac{\pi}{2}\right)\right) = \frac{\pi}{4}(0 + 1) = 0.785398$

$M_1 = \left(\frac{\pi}{2}\right) f\left(\frac{\pi}{4}\right) = \left(\frac{\pi}{2}\right) 0.840896 = 1.320876$

$S_2 = \frac{2}{3} M_1 + \frac{1}{3} T_1 = 1.142383$

$T_2 = \frac{T_1 + M_1}{2} = 1.053137$

$M_2 = \left(\frac{\pi}{4}\right)\left(f\left(\frac{\pi}{8}\right) + f\left(\frac{3\pi}{8}\right)\right) = \left(\frac{\pi}{4}\right)(0.618614 + 0.961187)$

$= 1.240773$

$S_4 = \frac{2}{3} M_2 + \frac{1}{3} T_2 = 1.178228$

$T_4 = \frac{T_2 + M_2}{2} = 1.146955$

$M_4 = \left(\frac{\pi}{8}\right)\left(f\left(\frac{\pi}{16}\right) + f\left(\frac{3\pi}{16}\right) + f\left(\frac{5\pi}{16}\right) + f\left(\frac{7\pi}{16}\right)\right)$

$= \left(\frac{\pi}{8}\right)(0.441690 + 0.745366 + 0.911850 + 0.990346)$

$= 1.213146$

$S_8 = \frac{2}{3} M_4 + \frac{1}{3} T_4 = 1.191082$

$T_8 = \frac{T_4 + M_4}{2} = 1.180051$

n	T_n	M_n	S_n
1	0.785398	1.320876	
2	1.053137	1.240773	1.142383
4	1.146955	1.213146	1.178228
8	1.180051		1.191082

Use the following Trapezium Rule estimates to obtain extrapolated values for T_{16} and T_∞.

n	T_n
1	0.785398
2	1.053137
4	1.146955
8	1.180051

Solution

n	T_n	$T_n - T_{\frac{n}{2}}$	Ratios
1	0.785398		
2	1.053137	0.267739	
4	1.146955	0.093818	0.350408
8	1.180051	0.033096	0.352768

[The values of k that are actually realised for the integration methods are often significantly different from the theoretical ones, and can be higher or lower.]

$T_{16} \approx T_8 + 0.35(T_8 - T_4) = 1.191635$
$T_\infty \approx T_8 + \frac{0.35}{1-0.35}(T_8 - T_4) = 1.197872$
Estimate for T: 1.20 (2dp) looks secure.

$\int_0^1 \sqrt{x}\ dx$ is to be estimated. Complete the following table, by the quickest method. Give values to 6 dp.

n	T_n	M_n	S_n
1			
2			
4			
8			

$\int_0^1 \sqrt{x}\ dx$ is to be estimated. Complete the following table, by the quickest method. Give values to 6 dp.

Solution

$T_1 = \frac{1}{2}(1)\big(f(0) + f(1)\big) = \frac{1}{2}(0 + 1) = 0.500000$
$M_1 = (1)f(0.5) = 0.707107$
$S_2 = \frac{2}{3}M_1 + \frac{1}{3}T_1 = 0.638071$
$T_2 = \frac{T_1 + M_1}{2} = 0.603554$
$M_2 = \left(\frac{1}{2}\right)\big(f(0.25) + f(0.75)\big) = \left(\frac{1}{2}\right)(0.500000 + 0.866025)$
$= 0.683013$
$S_4 = \frac{2}{3}M_2 + \frac{1}{3}T_2 = 0.656527$
$T_4 = \frac{T_2 + M_2}{2} = 0.643284$
$M_4 = \left(\frac{1}{4}\right)\big(f(0.125) + f(0.375) + f(0.625) + f(0.875)\big)$
$= \left(\frac{1}{4}\right)(0.353553 + 0.612372 + 0.790569 + 0.935414)$
$= 0.672977$
$S_8 = \frac{2}{3}M_4 + \frac{1}{3}T_4 = 0.663079$
$T_8 = \frac{T_4 + M_4}{2} = 0.658131$

$[\int_0^1 \sqrt{x}dx = \left[\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1 = \frac{2}{3}]$

n	T_n	M_n	S_n
1	0.500000	0.707107	
2	0.603554	0.683013	0.638071
4	0.643284	0.672977	0.656527
8	0.658131		0.663079

Using the S_n given, complete the following table of ratios of differences (where S is the exact value of $\frac{2}{3}$).

n	S_n	$S_n - S_{\frac{n}{2}}$	Ratios	$S_n - S$	Ratios
1					
2	0.638071				
4	0.656527				
8	0.663079				

Solution

n	S_n	$S_n - S_{\frac{n}{2}}$	Ratios	$S_n - S$	Ratios
1					
2	0.638071			-0.028596	
4	0.656527	0.018456		-0.010140	0.354595
8	0.663079	0.006552	0.355007	-0.003588	0.353846

[The values of k that are actually realised for the integration methods are often significantly different from the theoretical ones, and can be higher or lower.]

Solution of Equations

Describe the relative merits of the Secant method and the method of False Position.

Describe the relative merits of the Secant method and the method of False Position.

Solution

(i) For the method of False Position, a change of sign has to be investigated at each step (programming complication); for the Secant method, a sequence of estimates is produced automatically (and there needn't be a change of sign at any stage).

(ii) The method of False Position automatically produces an interval for the estimate; for the Secant method, an interval would have to be found by establishing a change of sign.

(iii) Provided the function is continuous, the method of False Position will always lead to a root (as it is trapped between bounds that get closer and closer); for the Secant method, the root isn't necessarily trapped, and convergence isn't guaranteed.

(iv) Convergence is usually faster for the Secant method than for the method of False Position.