Moments of Inertia (7 pages; 1/8/23)

(1) Suppose that an object rotates about an axis AB with angular velocity $\dot{\theta}$.

Considering the object to be made up of particles, suppose that particle i (with mass m_i) is at a perpendicular distance p_i from AB. Then the total kinetic energy of the particles is

$\sum_i \frac{1}{2} m_i (\dot{\theta} p_i)^2 = \frac{1}{2} I_{AB} (\dot{\theta})^2$ (as $\dot{\theta}$ is the same for all particles)

where $I_{AB} = \sum_i m_i (p_i)^2$ is defined to be the moment of inertia of the object about AB.

Thus, by comparison with (linear) kinetic energy, the moment of inertia takes the place of mass, whilst the angular velocity takes the place of (linear) velocity. Just as mass measures the resistance of an object to linear acceleration, the moment of inertia measures the resistance to rotational acceleration about a particular axis.

(2) Moments of inertia can be determined by integration (see proofs below).

Commonly used moments of inertia are shown below.

[G is the centre of mass; the objects are uniform]

	axis	
rod of length $2a$	through G, perpendicular to the rod	$\frac{1}{3}Ma^2$
rectangular lamina of sides $2a$ and $2b$	in plane of lamina, bisecting the side of length $2a$	$\frac{1}{3}Ma^2$
circular hoop of radius a	axis of symmetry	Ma^2
circular disc of radius a	axis of symmetry	$\frac{1}{2}Ma^2$
Hollow cone (with no base) of radius a	axis of the cone	$\frac{1}{2}Ma^2$
spherical shell of radius a	any axis through G	$\frac{2}{3}Ma^2$
solid sphere of radius a	any axis through G	$\frac{2}{5}Ma^2$

(3) Proofs

(i) Uniform rod of length $2a$, about an axis through G, perpendicular to the rod

$I = \int_{-a}^{a} x^2 (\rho dx)$,

where $\rho = \frac{M}{2a}$ is the density of the rod per unit length

So $I = \frac{M}{2a} \left[\frac{1}{3}x^3 \right]_{-a}^{a} = \frac{M}{6a}(2a^3) = \frac{1}{3}Ma^2$

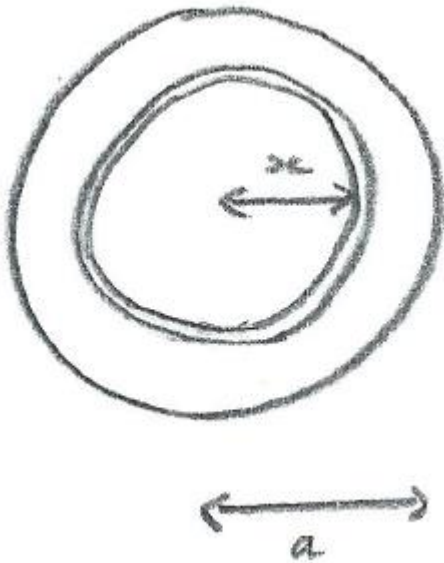(ii) Uniform circular disc of radius a, about the axis of symmetry

Figure 1

Consider a hoop element of radius x, as in Figure 1.

Its mass is $\sigma(2\pi x \delta x)$,

where $\sigma = \dfrac{M}{\pi a^2}$ is the density of the disc per unit area.

Then $I = \int_0^a x^2 (\, 2\sigma\pi x dx) = \dfrac{2M}{a^2} \int_0^a x^3 \, dx$

$= \dfrac{2M}{a^2} \left[\dfrac{1}{4} x^4 \right]_0^a = \dfrac{1}{2} M a^2$

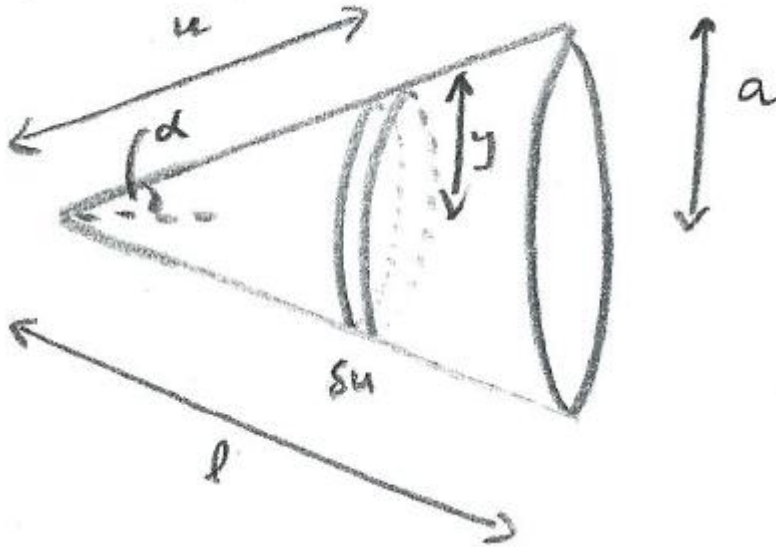(iii) Hollow cone (with no base) about its axis

Figure 2

The surface area of the element shown is $(2\pi y)\delta u$

The surface area of the cone is $\pi a l$,

so the density of the hollow cone is $\sigma = \dfrac{M}{\pi a l}$ per unit area,

and the mass of the element is $(2\pi y)(\delta u)\sigma = (2\pi y)(\delta u)\left(\dfrac{M}{\pi a l}\right)$

$= \dfrac{2yM}{al}\delta u$

Also, $y = u\sin\alpha$ and $a = l\sin\alpha$

$I = \int_0^l y^2 . \dfrac{2yM}{al} du = \dfrac{2M}{al} \int_0^l u^3 \sin^3\alpha \; du$

$= \dfrac{2M\sin^3\alpha}{al}\left[\dfrac{1}{4}u^4\right]_0^l = \dfrac{Ml^3\sin^3\alpha}{2a} = \dfrac{Ma^2}{2}$

(4) The moment of inertia of a body will be unaffected if each particle making up the body is moved parallel to the specified axis.

This means that the following pairs of bodies have the same moments of inertia:

rod of length $2a$	through G, perpendicular to the rod	$\frac{1}{3}Ma^2$
rectangular lamina of sides $2a$ and $2b$	in plane of lamina, bisecting the side of length $2a$	$\frac{1}{3}Ma^2$

circular hoop of radius a	axis of symmetry	Ma^2
cylindrical shell of radius a	axis of symmetry	Ma^2

circular disc of radius a	axis of symmetry	$\frac{1}{2}Ma^2$
cylinder of radius a	axis of symmetry	$\frac{1}{2}Ma^2$

(5) By symmetry, the formula for the moment of inertia of a hemisphere will be the same as that of a sphere (but with M now being the mass of the hemisphere).

(6) Perpendicular Axis Theorem (for a lamina): $I_z = I_x + I_y$

Considering a lamina in the x-y plane, let its moments of inertia about the x & y axes be I_x & I_y, respectively.

Then, if r is the distance of any point in the lamina from O,

$r^2 = x^2 + y^2$, and so the moment of inertia about the z-axis will be $I_x + I_y$. (This will be true whether the lamina is uniform or not.)

Examples

(i) To find the moment of inertia of a circular disc of radius a about a diameter:

Let the Origin be the centre of the disc. Then the moment of inertia about the axis of symmetry, $I_z = \frac{1}{2}Ma^2$ (from the table of standard results), and the moment of inertia about a diameter is $I_x = I_y$. Then, by the perpendicular axis theorem, $I_z = I_x + I_y$,

so that $I_x = \frac{1}{4}Ma^2$

(ii) To find the moment of inertia I of a rectangular lamina of sides $2a$ and $2b$, about the perpendicular axis through the centre:

The moment of inertia about the axis in the plane of the lamina, bisecting the side of length $2a$, is $\frac{1}{3}Ma^2$, so that, by the perpendicular axis theorem, $I = \frac{1}{3}Ma^2 + \frac{1}{3}Mb^2 = \frac{1}{3}M(a^2 + b^2)$

(7) Parallel Axis Theorem: $I_A = I_G + Md^2$

Let I_G be the moment of inertia of a rigid body about an axis that passes through the centre of gravity G, and let I_A be the moment of inertia about a parallel axis that passes through the point A, such that the shortest ('perpendicular') distance between the axes is $AG = d$.

Define the z-axis of a coordinate system to be the above-mentioned axis passing through G, and define the x-axis so that AG lies on it. Thus G is the Origin of the system.
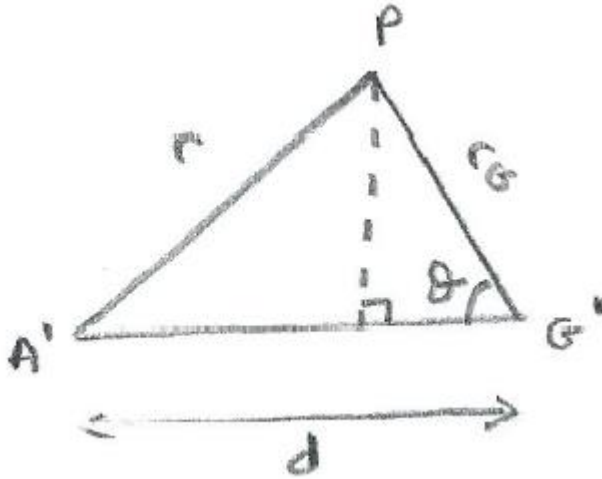
Figure 3

Referring to Figure 3, if P is the position of any particle in the body, the triangle $PA'G'$ can be constructed, where r and r_G are the 'perpendicular' (or shortest) distances of P from the two axes. The triangle will be in a plane parallel to the x-y plane.

By the cosine rule, $r^2 = r_G{}^2 + d^2 - 2dr_G cos\theta$

and $m_P r^2 = m_P r_G{}^2 + m_P d^2 - 2m_P dr_G cos\theta$, where m_P is the mass of particle P.

Then note that $r_G cos\theta = x_P$, the x coordinate of P, and as G is the Origin, $\sum m_P x_P = 0$, by definition of the centre of mass.

So, on summing over all particles, we obtain

$$I_A = I_G + Md^2 - 2d \sum m_P x_P = I_G + Md^2$$

Example

To find the moment of inertia I of a rod of length $2a$ about a perpendicular axis through one end of the rod:

As the moment of inertia through G, perpendicular to the rod, is $\frac{1}{3}Ma^2, I = \frac{1}{3}Ma^2 + Ma^2 = \frac{4}{3}Ma^2$