Matrices: Transformations – Exercises (with Sol'ns)

(14 pages ; 3/6/26)

(1) The point P is transformed by the matrix $\begin{pmatrix} 1 & 1 & 2 \\ -2 & 3 & 0 \\ 0 & 4 & -1 \end{pmatrix}$ to the image point (4, 2, –3). Find the coordinates of P.

[without using a calculator]

The point P is transformed by the matrix $\begin{pmatrix} 1 & 1 & 2 \\ -2 & 3 & 0 \\ 0 & 4 & -1 \end{pmatrix}$

to the image point (4, 2, –3). Find the coordinates of P.

[without using a calculator]

Solution

Let $A = \begin{pmatrix} 1 & 1 & 2 \\ -2 & 3 & 0 \\ 0 & 4 & -1 \end{pmatrix}$

Then $AP = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$, [1 mark]

and $P = A^{-1}\begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$ [1 mark]

$|A| = 1(-3) - (-2)(-9) = -21$ [1 mark]

and $A^{-1} = -\frac{1}{21}\begin{pmatrix} -3 & -2 & -8 \\ 9 & -1 & -4 \\ -6 & -4 & 5 \end{pmatrix}^T$ [3 marks]

so that $P = -\frac{1}{21}\begin{pmatrix} -3 & 9 & -6 \\ -2 & -1 & -4 \\ -8 & -4 & 5 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} = -\frac{1}{21}\begin{pmatrix} 24 \\ 2 \\ -55 \end{pmatrix}$

The coordinates of P are $(-\frac{8}{7}, -\frac{2}{21}, \frac{55}{21})$ [2 marks]

(2) $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ represents a transformation.

(i) Under what conditions will $x = 0$ be an invariant line?

(ii) Under what conditions will there be an invariant line of the form $x = \lambda$ (where $\lambda \neq 0$)?

$M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ represents a transformation.

(i) Under what conditions will $x = 0$ be an invariant line?

(ii) Under what conditions will there be an invariant line of the form $x = \lambda$ (where $\lambda \neq 0$)?

Solution

(i) Suppose that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y' \end{pmatrix}$, for all y.

Then $cy = 0$ for all y, so that $c = 0$

(ii) Suppose that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} \lambda \\ y \end{pmatrix} = \begin{pmatrix} \lambda \\ y' \end{pmatrix}$, for all y.

Then $a\lambda + cy = \lambda$ for all y, so that $c = 0$, and $a = 1$

(3)(i) Plot the image of the unit square under the transformation represented by the matrix $\begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix}$

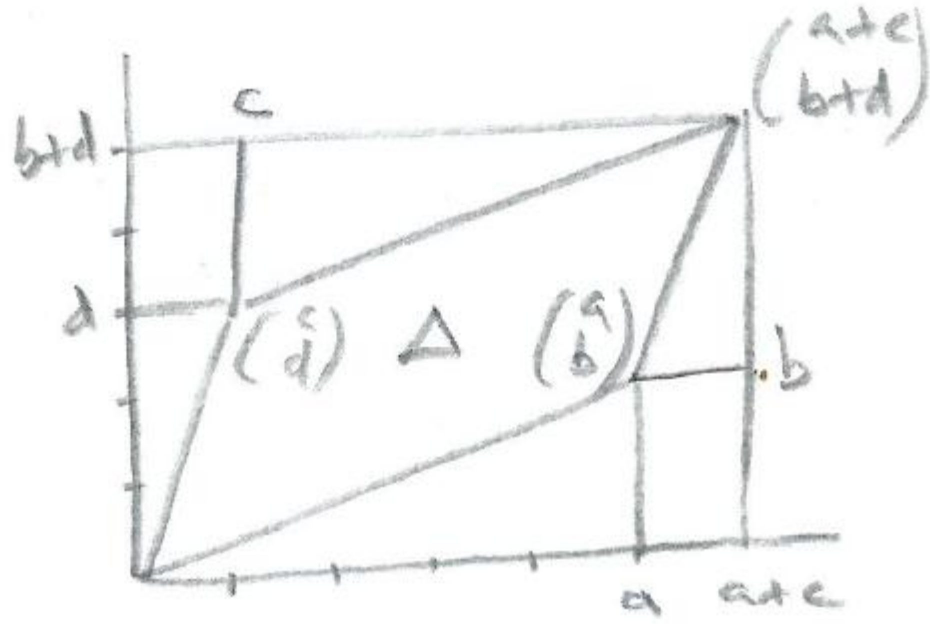
(ii) Use (i), but with more general labels, to show that the area scale factor for the transformation $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is $ad - bc$.

(i) Plot the image of the unit square under the transformation represented by the matrix $\begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix}$

(ii) Use (i), but with more general labels, to show that the area scale factor for the transformation $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is $ad - bc$.

Solution

(ii)



$\Delta = (a + c)(b + d) - 2(\frac{1}{2}cd + bc + \frac{1}{2}ab)$

$= ab + ad + cb + cd - cd - 2bc - ab$

$= ad - bc$

(4)(i) Find the equation of the line that the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ maps all points to.

(ii) For the same transformation, find the equation of the line that maps to the point with an x-coordinate of w.

(iii) For the same transformation, for which point(s) will the x-coordinate remain unchanged by the transformation?

(i) Find the equation of the line that the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ maps all points to.

(ii) For the same transformation, find the equation of the line that maps to the point with an x-coordinate of w.

(iii) For the same transformation, for which point(s) will the x-coordinate remain unchanged by the transformation?

Solution

(i) $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$

$\Rightarrow p + 2q = u$ and $3p + 6q = v$

so that $v = 3u$; equation of line is $y = 3x$

(ii) $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} w \\ 3w \end{pmatrix} \Rightarrow p + 2q = w$

$\Rightarrow q = \frac{1}{2}(w - p)$

ie equation is $y = \frac{w}{2} - \frac{x}{2}$

(iii) $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ 3p \end{pmatrix} \Rightarrow p + 2q = p$ (so that $q = 0$),

(and $3p + 6q = 3p$)

Thus all points on the x-axis map to points with the same x-coordinate.

(5) Derive a formula for the area of a triangle with corners at $(0,0), (a, b), (c, d)$, using matrix transformations.

Derive a formula for the area of a triangle with corners at $(0,0), (a,b), (c,d)$, using matrix transformations.

Solution

The formula for the area of a triangle with corners $(0,0), (a,b), (c,d)$ can be obtained by considering the matrix transformation $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$: (a,b) is the image of (1,0) and (c,d) is the image of (0,1); the area of the triangle with corners (0,0), (1,0), (0,1) is $\frac{1}{2}$, and the area scale factor is $|ad-bc|$, since $ad-bc$ is the determinant of the matrix (the modulus sign only being needed when the order of the corners becomes reversed in the course of the transformation). So the required area is $\frac{1}{2}|ad-bc|$.

(6) Show that the matrix $\begin{pmatrix} cos2\theta & sin2\theta \\ sin2\theta & -cos2\theta \end{pmatrix}$ [representing a reflection in the line $y = tan\theta . x$] can be written as $\begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{pmatrix}$

, where $m = tan\theta$

Show that the matrix $\begin{pmatrix} cos2\theta & sin2\theta \\ sin2\theta & -cos2\theta \end{pmatrix}$ [representing a reflection in the line $y = tan\theta . x$] can be written as $\begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{pmatrix}$, where $m = tan\theta$

Solution

$tan2\theta = \frac{2tan\theta}{1-tan^2\theta} = \frac{2m}{1-m^2}$

The right-angled triangle with opposite and adjacent sides of $2m$ & $1 - m^2$ has a hypotenuse of $\sqrt{4m^2 + (1 - 2m^2 + m^4)}$

$= \sqrt{(1 + m^2)^2} = 1 + m^2$,

so that $sin2\theta = \frac{2m}{1+m^2}$ & $cos2\theta = \frac{1-m^2}{1+m^2}$, as required.

(7)(i) Show that the transformation represented by the matrix $\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$ (with determinant zero) maps all points to a particular line.

(ii) Find the line whose points all map to the point $(3,1)$.

(iii) Without doing any calculations, what can be said about the line whose points all map to the point $(6,2)$?

(iv) Write down the line whose points all map to the Origin.

(i) Show that the transformation represented by the matrix $\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$ (with determinant zero) maps all points to a particular line.

(ii) Find the line whose points all map to the point (3,1).

(iii) Without doing any calculations, what can be said about the line whose points all map to the point (6,2)?

(iv) Write down the line whose points all map to the Origin.

Solution

(i) $\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x+6y \\ x+2y \end{pmatrix} = \begin{pmatrix} 3(x+2y) \\ x+2y \end{pmatrix}$

So all points map to the line $y = \frac{1}{3}x$

(ii) $\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow x + 2y = 1$

ie the required line is $y = -\frac{1}{2}x + \frac{1}{2}$

(iii) The line will have gradient $-\frac{1}{2}$ (lines mapping to different points cannot intersect (and therefore must be parallel) - otherwise the intersection point would map to two points).

[Note: The line doesn't contain the point (6,2).]

(iv) $y = -\frac{1}{2}x$ (It must contain the Origin, as this always maps to itself.)