Matrices: Invariant Points & Lines – Exercises (with Sol'ns)(39 pages ; 3/6/26)

(1) Find the value of k for which the transformation $\begin{pmatrix} 2 & 4 \\ 3 & k \end{pmatrix}$ has a line of invariant points, and find this line.

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Solution

Suppose that $\begin{pmatrix} 2 & 4 \\ 3 & k \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$

Then $2p + 4q = p$ and $3p + kq = q$

so that $4q = -p$ & $(k-1)q = -3p$

Hence $\frac{q}{p} = -\frac{1}{4}$ & $\frac{q}{p} = -\frac{3}{k-1}$

so that $-\frac{1}{4} = -\frac{3}{k-1} \Rightarrow k - 1 = 12$ & hence $k = 13$

[See "Invariant Points & Lines - Conditions". A line of invariant points will exist when $trM = |M| + 1$; in this case, when $2 + k = 2k - 12 + 1$]

To find the line:

$\begin{pmatrix} 2 & 4 \\ 3 & 13 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \Rightarrow 2p + 4q = p$ & $3p + 13q = q$

so that $4q = -p$ (or $12q = -3p$)

and hence $q = -\frac{p}{4}$

ie the invariant points lie on the line $y = -\frac{x}{4}$

Check: $\begin{pmatrix} 2 & 4 \\ 3 & 13 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

(2)(i) Use a matrix method to find the invariant lines for a reflection in the y-axis.

(ii) Investigate the invariant lines for a reflection in the x-axis.

(i) Use a matrix method to find the invariant lines for a reflection in the y-axis.

(ii) Investigate the invariant lines for a reflection in the x-axis.

Solution

(i) Suppose that an invariant line has the equation $y = mx + c$ (noting that lines of the form $x = a$ aren't invariant lines)

The image of a point on this line is:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} -x \\ mx + c \end{pmatrix}$$

For this image to lie on the line, we require that

$m(-x) + c = mx + c$

$\Rightarrow 2mx = 0$

$\Rightarrow m = 0$ (for any value of c), or $x = 0$

ie the invariant lines are $y = c$ and $x = 0$ (the line of invariant points)

(ii) Suppose that an invariant line has the equation $y = mx + c$.

The image of a point on this line is:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} x \\ -mx - c \end{pmatrix}$$

For this image to lie on the line, we require that

$mx + c = -mx - c$

Equating coefficients of x: $m = -m \Rightarrow m = 0$

Equating the constant terms: $c = -c \Rightarrow c = 0$

So we have only found the line $y = 0$ (the line of invariant points).

Now consider lines of the form $x = a$.

This gives $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ y \end{pmatrix} = \begin{pmatrix} a \\ -y \end{pmatrix}$

As this lies on the line $x = a$ for all values of a, the lines $x = a$ are also invariant lines.

(3) $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ represents a transformation.

(i) Under what conditions will $x = 0$ be an invariant line?

(ii) Under what conditions will there be an invariant line of the form $x = \lambda$ (where $\lambda \neq 0$)?

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Solution

(i) Suppose that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y' \end{pmatrix}$, for all y.

Then $cy = 0$ for all y,

so that $c = 0$

(ii) Suppose that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} \lambda \\ y \end{pmatrix} = \begin{pmatrix} \lambda \\ y' \end{pmatrix}$, for all y.

Then $a\lambda + cy = \lambda$ for all y,

so that $c = 0$, and $a = 1$

(4a) $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ represents a transformation.

Under what conditions will there be a line of invariant points passing through the Origin?

[It can in fact be shown that any line of invariant points will pass through the Origin.]

$M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ represents a transformation.

Under what conditions will there be a line of invariant points passing through the Origin?

Solution

Suppose that there is a line of invariant points $y = mx$,

so that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} x \\ mx \end{pmatrix}$ for all x

ie $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix}$

or $\begin{pmatrix} a-1 & c \\ b & d-1 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

For there to be a solution other than $x = 0, y = 0$,

$\begin{vmatrix} a-1 & c \\ b & d-1 \end{vmatrix} = 0$

$\Rightarrow (a-1)(d-1) - bc = 0$

$\Rightarrow 1 - (a+d) + ad - bc = 0$

$\Rightarrow trM = |M| + 1$

(4b) A transformation is represented by $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, where

$c \neq 0$

Show that the condition for the existence of a line of invariant

points is that $tr(M) = |M| + 1$

[The trace of M, $tr(M) = a + d$.]

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points is that $tr(M) = |M| + 1$

[The trace of M, $tr(M) = a + d$.]

Solution

Suppose that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$

Then $ap + cq = p$ & $bp + dq = q$,

so that $(a-1)p + cq = 0$ & $bp + (d-1)q = 0$

In order for there to be a solution other than $p = q = 0$,

$\begin{vmatrix} a-1 & c \\ b & d-1 \end{vmatrix} = 0$,

so that $(a-1)(d-1) - bc = 0$,

and $ad - bc - (a+d) + 1 = 0$;

giving $tr(M) = \mathrm{a} + \mathrm{d} = |M| + 1$

This argument is reversible, and so there will be a line of invariant

points when $tr(M) = |M| + 1$

(5) Find the equations of the invariant lines of the transformation represented by the matrix $\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}$

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Solution

[Note that the transformation is a shear, as the determinant is 1, and the trace $(4 + (-2))$ equals 2. Note also that, in general, the eigenvectors of a matrix give the invariant lines that pass through the Origin. In the case of a shear, which has repeated eigenvalues of 1 (see "Matrices - Notes"), the single eigenvector is the line of invariant points, and the other invariant lines will be parallel to this.]

Suppose that $\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} x' \\ mx' + c \end{pmatrix}$ for all x

Then $4x + 3(mx + c) = x'$

and $-3x - 2(mx + c) = mx' + c$

$\Rightarrow -3x - 2mx - 2c = m(4x + 3mx + 3c) + c$

$\Rightarrow x(-3 - 2m - 4m - 3m^2) - 2c - 3mc - c = 0$

$\Rightarrow x(3m^2 + 6m + 3) + 3c + 3mc = 0$

$\Rightarrow x(m^2 + 2m + 1) + c(1 + m) = 0$

Equating coeffs of powers of x,

$m^2 + 2m + 1 = 0 \Rightarrow (m + 1)^2 = 0 \Rightarrow m = -1$

and either $c = 0$ or $m = -1$

Thus $m = -1$ and c can take any value,

so that the invariant lines are $y = -x + c$

(6) Prove that invariant points of a 2D transformation always lie on a line passing through the Origin.

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Solution

Suppose that $y = px + q$ is a line of invariant points for the transformation $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

Then $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ px+q \end{pmatrix} = \begin{pmatrix} x \\ px+q \end{pmatrix}$

$\Rightarrow ax + cpx + cq = x$

and $bx + dpx + dq = px + q$

As these equations are to hold for any x, we can equate coefficients of x and the constant terms to give:

$a + cp = 1$ (1)

$cq = 0$ (2)

$b + dp = p$ (3)

$dq = q$ (4)

Suppose that $q \neq 0$

Then (2) & (4) $\Rightarrow c = 0$ & $d = 1$

and (1) & (3) $\Rightarrow a = 1$ & $b = 0$

ie $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, in which case all points are invariant.

Thus, apart from this trivial case, $q = 0$ and so a line of invariant points takes the form $y = px$, and therefore passes through the Origin.

(7) For the transformation matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$, where a, b, c & d are positive, find a relationship between the trace $a + d$ and the determinant that must hold in order for the transformation to have an invariant line that doesn't pass through the Origin.

For the transformation matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$, where a, b, c & d are positive, find a relationship between the trace $a + d$ and the determinant that must hold in order for the transformation to have an invariant line that doesn't pass through the Origin.

Solution

Suppose that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ mx + k \end{pmatrix} = \begin{pmatrix} x' \\ mx' + k \end{pmatrix}$ for all x,

where $k \neq 0$

Then $ax + cmx + ck = x'$ & $bx + dmx + dk = mx' + k$

and $(a + cm)x + ck = x'$ & $(b + dm)x + (d - 1)k = mx'$

Multiplying the 1st equation by m and equating the two expressions for mx' gives:

$m(a + cm)x + mck = (b + dm)x + (d - 1)k$

As this is to hold for all x, we can equate the coefficients of x , to give:

$m(a + cm) = b + dm$ & $mck = (d - 1)k \quad (1)$

Thus, as k & $c \neq 0$,

$cm^2 + (a - d)m - b = 0$ & $m = \frac{d-1}{c}$,

and hence $c\left(\frac{d-1}{c}\right)^2 + (a - d)\left(\frac{d-1}{c}\right) - b = 0$,

so that $(d - 1)^2 + (a - d)(d - 1) - bc = 0$

and $(d - 1)(d - 1 + a - d) - bc = 0$,

giving $(d - 1)(a - 1) - bc = 0$

and hence $ad - bc - (a + d) + 1 = 0$

or $trace = det + 1$

(8) For the matrix $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$, find all of the invariant lines of the associated transformation.

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Solution

Suppose that $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} x' \\ mx' + c \end{pmatrix}$ for all x,

Then $2x + 2mx + 2c = x'$ & $x + 3mx + 3c = mx' + c$

and $(2 + 2m)x + 2c = x'$ & $(1 + 3m)x + 2c = mx'$

Multiplying the 1st equation by m and equating the two expressions for mx' gives:

$m(2 + 2m)x + 2mc = (1 + 3m)x + 2c$

As this is to hold for all x, we can equate the coefficients of x , to give:

$m(2 + 2m) = 1 + 3m$ & $2mc = 2c$ (1)

Case 1: $c = 0$ (invariant lines through the Origin; ie the eigenvectors)

$2m^2 - m - 1 = 0 \Rightarrow (2m + 1)(m - 1) = 0$,

so that $m = -\frac{1}{2}$ & 1

[Check: To find the eigenvalues of the matrix, we create the characteristic equation $\begin{vmatrix} 2 - \lambda & 2 \\ 1 & 3 - \lambda \end{vmatrix} = 0$

$\Rightarrow (2 - \lambda)(3 - \lambda) - 2 = 0$

and $\lambda^2 - 5\lambda + 4 = 0$, so that $\lambda_1 = 1$ & $\lambda_2 = 4$

Then, to find the eigenvectors:

$\begin{pmatrix} 2 - 1 & 2 \\ 1 & 3 - 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so that $x + 2y = 0$; ie one invariant line is $y = -\frac{1}{2}x$

and $\begin{pmatrix} 2-4 & 2 \\ 1 & 3-4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

so that $x - y = 0$; ie another invariant line is $y = x$

(and these agree with the values of m above)]

Case 2: $c \neq 0$

From (1), $m = 1$

Thus, the other invariant lines are those of the form $y = x + c$

(9) Find the invariant points and lines for the transformation represented by the matrix $\begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}$.

Find the invariant points and lines for the transformation represented by the matrix $\begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}$.

Solution

Invariant points satisfy $\begin{pmatrix} \mathbf{5} & \mathbf{4} \\ \mathbf{-4} & \mathbf{-3} \end{pmatrix}\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}$

$\Rightarrow \mathbf{5}\boldsymbol{x} + \mathbf{4}\boldsymbol{y} = \boldsymbol{x}, -\mathbf{4}\boldsymbol{x} - \mathbf{3}\boldsymbol{y} = \boldsymbol{y} \Rightarrow \boldsymbol{y} = -\boldsymbol{x}$

This is a line of invariant points.

For points on the line $\boldsymbol{y} = \boldsymbol{mx} + \boldsymbol{c}$,

$\begin{pmatrix} \mathbf{5} & \mathbf{4} \\ \mathbf{-4} & \mathbf{-3} \end{pmatrix}\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{mx} + \boldsymbol{c} \end{pmatrix} = \begin{pmatrix} \mathbf{5}\boldsymbol{x} + \mathbf{4}\boldsymbol{mx} + \mathbf{4}\boldsymbol{c} \\ -\mathbf{4}\boldsymbol{x} - \mathbf{3}\boldsymbol{mx} - \mathbf{3}\boldsymbol{c} \end{pmatrix}$

For this to be an invariant line,

$-\mathbf{4}\boldsymbol{x} - \mathbf{3}\boldsymbol{mx} - \mathbf{3}\boldsymbol{c} = \boldsymbol{m}(\mathbf{5}\boldsymbol{x} + \mathbf{4}\boldsymbol{mx} + \mathbf{4}\boldsymbol{c}) + \boldsymbol{c}$ for all $\boldsymbol{x}$

Equating coefficients of $\boldsymbol{x}$, $-\mathbf{4} - \mathbf{3}\boldsymbol{m} = \mathbf{5}\boldsymbol{m} + \mathbf{4}\boldsymbol{m}^{\mathbf{2}}$,

so that $\mathbf{4}\boldsymbol{m}^{\mathbf{2}} + \mathbf{8}\boldsymbol{m} + \mathbf{4} = \mathbf{0}$, or $\boldsymbol{m}^{\mathbf{2}} + \mathbf{2}\boldsymbol{m} + \mathbf{1} = \mathbf{0}$;

ie $(\boldsymbol{m} + \mathbf{1})^{\mathbf{2}} = \mathbf{0}$, so that $\boldsymbol{m} = -\mathbf{1}$

Equating the constant terms, $-\mathbf{3}\boldsymbol{c} = \mathbf{4}\boldsymbol{mc} + \boldsymbol{c}$

$\Rightarrow \boldsymbol{c}(\mathbf{4}\boldsymbol{m} + \mathbf{4}) = \mathbf{0}$

$\Rightarrow \boldsymbol{c} = \mathbf{0}$ or $\boldsymbol{m} = -\mathbf{1}$

So the overall condition is: $\boldsymbol{m} = -\mathbf{1}$ and $\boldsymbol{c}$ can take any value,

and the invariant lines are of the form $\boldsymbol{y} = \boldsymbol{c} - \boldsymbol{x}$ (including the line of invariant points $\boldsymbol{y} = -\boldsymbol{x}$).

[Note: $\begin{pmatrix} \mathbf{5} & \mathbf{4} \\ \mathbf{-4} & \mathbf{-3} \end{pmatrix}$ represents a shear, as its determinant is 1 and the sum of the elements on the leading diagonal is 2. It follows that the invariant lines will all be parallel to the line of invariant points.]

(10) For the transformation $\begin{pmatrix} 3 & 1 \\ 2 & k \end{pmatrix}$:

(i) For what value of k is there a line of invariant points, and find this line.

(ii) Given now that $k = 4$, find any invariant lines of the transformation.

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(i) For what value of k is there a line of invariant points, and find this line.

(ii) Given now that $k = 4$, find any invariant lines of the transformation.

Solution

(i) Suppose that $\begin{pmatrix} 3 & 1 \\ 2 & k \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$

Then $3p + q = p$ and $2p + kq = q$

so that $q = -2p$ & $2p = q(1 - k)$

Hence $\frac{q}{p} = -2$ & $\frac{q}{p} = \frac{2}{1-k}$

so that $-2 = \frac{2}{1-k} \Rightarrow 1 - k = -1$ & hence $k = 2$

As $q = -2p$, the invariant points lie on the line $y = -2x$

[Check: $\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$]

(ii) Suppose that there is an invariant line $y = mx + c$.

Considering a point on this line:

$$\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} 3x + mx + c \\ 2x + 4(mx + c) \end{pmatrix}$$

And, for the line to be invariant,

$2x + 4(mx + c) = m(3x + mx + c) + c$ for all x

Equating coefficients of x: $2 + 4m = 3m + m^2$;

ie $m^2 - m - 2 = 0$

so that $(m - 2)(m + 1) = 0$, and hence $m = 2$ or -1

Equating the constant term: $4c = mc + c$;

ie $c = 0$ or $m = 3$

Combining these two conditions, $m = 2$ or -1, and $c = 0$

So the invariant lines are $y = 2x$ and $y = -x$

(11) The 2×2 matrix M represents a transformation that maps all points to the same line. In addition, points on this line are invariant points under the transformation. What conditions must apply to M?

The 2×2 matrix M represents a transformation that maps all points to the same line. In addition, points on this line are invariant points under the transformation. What conditions must apply to M?

Solution

As all points are mapped to the same line, $|M| = 0$.

Let $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, so that $|M| = ad - bc = 0$.

Let $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$

Then $ap + cq = u$ & $bp + dq = v$,

and as $\frac{b}{a} = \frac{d}{c}$ (from $ad - bc = 0$), it follows that $\frac{v}{u} = \frac{b}{a}$ also,

and the equation of the line to which all points map is $y = \frac{b}{a} .x$

It is given that all points on this line are invariant:

hence $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} p \\ \frac{b}{a} .p \end{pmatrix} = \begin{pmatrix} p \\ \frac{b}{a} .p \end{pmatrix}$,

so that $ap + c.\frac{b}{a} .p = p$, or $a + c.\frac{d}{c} = 1$

(as this must hold for all p)

ie $a + d$ (the trace of M) $= 1$

[The other equation also gives $bp + d.\frac{b}{a} .p = \frac{b}{a}.p \Rightarrow a + d = 1$]

Thus the required conditions are that $|M| = 0$ and $tr(M) = 1$.

(12) For a 2×2 matrix $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ with zero determinant:

(i) Find the equation of the line (which we will call the image line) to which all points are mapped.

(ii) Find the family of lines that map to specific points on the image line.

(iii) What condition must apply to a, b, c & d in order for the lines in this family to be perpendicular to the image line?

(iv) Show that if the lines in this family are invariant lines under the transformation, then $tr(M) = 1$, where $tr(M)$ (the trace of M) is defined to be $a + d$

(v) Find the condition(s) for the lines in this family to be invariant lines that are perpendicular to the image line, and give an example of a matrix satisfying these conditions.

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(v) Find the condition(s) for the lines in this family to be invariant lines that are perpendicular to the image line, and give an example of a matrix satisfying these conditions.

Solution

(i) Consider $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$

Then $ap + cq = u,\ \ bp + dq = v$

And, as the determinant is zero, $\frac{b}{a} = \frac{d}{c}$, so that $v = \frac{b}{a}u$;

ie the image line is $y = \frac{b}{a}x$

(ii) Consider a point on the image line with position vector $\begin{pmatrix} u \\ \frac{b}{a}u \end{pmatrix}$.

Then $ap + cq = u$

[$bp + dq = \frac{b}{a}u$ provides no further information]

so that $q = \frac{1}{c}(u - ap)$;

ie the family of lines is $y = -\frac{a}{c}x + \frac{u}{c}$

(iii) The lines in this family will be perpendicular to the image line

when $\left(-\frac{a}{c}\right).\left(\frac{b}{a}\right) = -1$, so that $b = c$

[$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ will then have the form $\begin{pmatrix} a & b \\ b & \frac{b^2}{a} \end{pmatrix}$]

(iv) In order for the lines in this family to be invariant lines, the

point $\begin{pmatrix} u \\ \frac{b}{a}u \end{pmatrix}$ has to lie on the original line (in the family);

ie $\frac{b}{a}u = -\frac{a}{c}u + \frac{u}{c}$

In order for this to hold for all u, $\frac{b}{a} = -\frac{a}{c} + \frac{1}{c}$

and $\frac{bc}{a} = 1 - a$

Then, as $|M| = 0 \Rightarrow \frac{b}{a} = \frac{d}{c}$, it follows that $d = 1 - a$,

and so $tr(M) = 1$, as required.

[In this situation, the image line will be a line of invariant points, and it can be shown that the general requirement for the existence of a line of invariant points is that $tr(M) = |M| + 1$.]

(v) Combining the conditions from (iii) and (iv),

$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ has the form $\begin{pmatrix} a & b \\ b & \frac{b^2}{a} \end{pmatrix}$ with $tr(M) = 1$, so that

$a + \frac{b^2}{a} = 1$, and hence $a^2 + b^2 = a$; $b^2 = a(1-a)$,

so that $b = \pm\sqrt{a(1-a)}$, with $a(1-a) \geq 0$; ie $0 \leq a \leq 1$

In conclusion:

$0 \leq a \leq 1$

$c = b = \pm\sqrt{a(1-a)}$

$d = 1 - a$

For example, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

[Projections of points onto the x and y axes respectively.]

or $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ or $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$

[$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(p+q) \\ \frac{1}{2}(p+q) \end{pmatrix}$, so that the image line (and line of

invariant points) is $y = x$; and a member of the family of lines

passes through $\begin{pmatrix} p \\ q \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{2}(p+q) \\ \frac{1}{2}(p+q) \end{pmatrix}$, so that its gradient is

$\frac{\frac{1}{2}(p+q)-q}{\frac{1}{2}(p+q)-p} = \frac{\frac{1}{2}(p-q)}{\frac{1}{2}(q-p)} = -1$, as expected.]

(13) Consider the transformation represented by the matrix

$M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

(i) Show that the transformation has at least one invariant line

if and only if $[tr(M)]^2 \geq 4|M|$

(where $tr(M)$ (the trace of M) $= a + d$)

(ii) Show that there will be a family of invariant lines if and only if

$tr(M) = |M| + 1$

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(where $tr(M)$ (the trace of M) $= a + d$)

(ii) Show that there will be a family of invariant lines if and only if

$tr(M) = |M| + 1$

Solution

(i) Suppose that there is an invariant line $y = mx + k$.

Then $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ mx + k \end{pmatrix} = \begin{pmatrix} ax + cmx + ck \\ bx + dmx + dk \end{pmatrix}$

and $bx + dmx + dk = m(ax + cmx + ck) + k$

This must apply for all x, so:

equating coefficients of x: $b + dm = ma + cm^2$,

so that $cm^2 + m(a - d) - b = 0$ (1)

And equating constant terms gives $dk = mck + k$,

so that $k(d - mc - 1) = 0$ (2)

Then, in order for there to be an m that satisfies (1),

the discriminant of (1) must be non-negative (and vice-versa);

ie $(a - d)^2 + 4cb \geq 0$

$\Leftrightarrow (a + d)^2 - 4ad + 4cb \geq 0$

$\Leftrightarrow [tr(M)]^2 \geq 4|M|$, as required.

(ii) From (2), if the condition in (i) is satisfied, there will only be one invariant line (with $k = 0$) unless $d - mc - 1 = 0$;

ie $m = \frac{d-1}{c}$

Then, from (1): $cm^2 + m(a - d) - b = 0$,

so that $(d - 1)^2 + (d - 1)(a - d) - bc = 0$

or $d^2 - 2d + 1 + da - d^2 - a + d - bc = 0$

or $-d + 1 + ad - a - bc = 0$

or $tr(M) = |M| + 1$, as required.

Then $[tr(M)]^2 - 4|M| = [|M| + 1]^2 - 4|M|$

$= [|M| - 1]^2 \geq 0$,

so that the condition in (i) is satisfied.

This argument is reversible, and so there will be a family of invariant lines if and only if $tr(M) = |M| + 1$.

(14) Suppose that the matrix $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ represents a reflection in the line $y = mx$.

(i) By considering the image of a particular point under M,

explain why $a^2 + b^2 = 1$.

(ii) Given that a necessary and sufficient condition for a

transformation to have a line of invariant points is that

$tr(M) = |M| + 1$ (where $tr(M)$ (the trace of M) $= a + d$)

and by considering the image of a point on the line $y = mx$ (or

otherwise), show that $M = \begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{pmatrix}$

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Solution

(i) $\begin{pmatrix} a \\ b \end{pmatrix}$ is the image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and the distance of $\begin{pmatrix} a \\ b \end{pmatrix}$ from the Origin

$(a^2 + b^2)$ will equal the distance of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ from the Origin (1);

thus $a^2 + b^2 = 1$

(ii) When any shape is reflected in the line $y = mx$, the area will

be unchanged, but the order of the vertices will be reversed (so

that clockwise becomes anti-clockwise). Hence $|M| = -1$.

Also, from the given result,

$a + d = |M| + 1 = -1 + 1 = 0$

So we can write $M = \begin{pmatrix} a & \frac{1-a^2}{b} \\ b & -a \end{pmatrix}$, where $a^2 + b^2 = 1$.

As $\begin{pmatrix} 1 \\ m \end{pmatrix}$ lies on the line of reflection,

$\begin{pmatrix} a & \frac{1-a^2}{b} \\ b & -a \end{pmatrix}\begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} 1 \\ m \end{pmatrix}$,

and so $a + \left(\frac{1-a^2}{b}\right)m = 1$, and $b - am = m$

Thus $b = m(a+1)$ [and $a + (1-a) = 1$]

Then $M = \begin{pmatrix} a & \frac{1-a^2}{m(a+1)} \\ m(a+1) & -a \end{pmatrix}$

and $a^2 + m^2(a+1)^2 = 1$,

so that $m^2 = \frac{1-a^2}{(1+a)^2} = \frac{1-a}{1+a}$;

$m^2 + am^2 = 1 - a$;

$a(m^2+1) = 1 - m^2$;

$a = \frac{1-m^2}{1+m^2}$

Then $b = m(a+1) = \frac{m(1-m^2)+m(1+m^2)}{1+m^2} = \frac{2m}{1+m^2}$

And $c = \frac{1-a^2}{m(a+1)} = \frac{(1-a)}{m} = \frac{1}{m}.\frac{(1+m^2)-(1-m^2)}{1+m^2} = \frac{2m}{1+m^2}$

So $M = \begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{pmatrix}$, as required.