Matrices: Invariant Points & Lines – Exercises

(4 pages ; 3/6/26)

(1) Find the value of k for which the transformation $\begin{pmatrix} 2 & 4 \\ 3 & k \end{pmatrix}$ has a line of invariant points, and find this line.

(2)(i) Use a matrix method to find the invariant lines for a reflection in the y-axis.

(ii) Investigate the invariant lines for a reflection in the x-axis.

(3) $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ represents a transformation.

(i) Under what conditions will $x = 0$ be an invariant line?

(ii) Under what conditions will there be an invariant line of the form $x = \lambda$ (where $\lambda \neq 0$)?

(4a) $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ represents a transformation.

Under what conditions will there be a line of invariant points passing through the Origin?

[It can in fact be shown that any line of invariant points will pass through the Origin.]

(4b) A transformation is represented by $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, where

$c \neq 0$

Show that the condition for the existence of a line of invariant

points is that $tr(M) = |M| + 1$

[The trace of M, $tr(M) = a + d$.]

(5) Find the equations of the invariant lines of the transformation represented by the matrix $\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}$

(6) Prove that invariant points of a 2D transformation always lie on a line passing through the Origin.

(7) For the transformation matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$, where a, b, c & d are positive, find a relationship between the trace $a + d$ and the

determinant that must hold in order for the transformation to

have an invariant line that doesn't pass through the Origin.

(8) For the matrix $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$, find all of the invariant lines of the associated transformation.

(9) Find the invariant points and lines for the transformation represented by the matrix $\begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}$.

(10) For the transformation $\begin{pmatrix} 3 & 1 \\ 2 & k \end{pmatrix}$:

(i) For what value of k is there a line of invariant points, and find this line.

(ii) Given now that $k = 4$, find any invariant lines of the transformation.

(11) The 2×2 matrix M represents a transformation that maps all points to the same line. In addition, points on this line are invariant points under the transformation. What conditions must apply to M?

(12) For a 2×2 matrix $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ with zero determinant:

(i) Find the equation of the line (which we will call the image line) to which all points are mapped.

(ii) Find the family of lines that map to specific points on the image line.

(iii) What condition must apply to a, b, c & d in order for the lines in this family to be perpendicular to the image line?

(iv) Show that if the lines in this family are invariant lines under the transformation, then $tr(M) = 1$, where $tr(M)$ (the trace of M) is defined to be $a + d$

(v) Find the condition(s) for the lines in this family to be invariant lines that are perpendicular to the image line, and give an example of a matrix satisfying these conditions.

(13) Consider the transformation represented by the matrix

$M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

(i) Show that the transformation has at least one invariant line

if and only if $[tr(M)]^2 \geq 4|M|$

(where $tr(M)$ (the trace of M) $= a + d$)

(ii) Show that there will be a family of invariant lines if and only if

$tr(M) = |M| + 1$

(14) Suppose that the matrix $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ represents a reflection in the line $y = mx$.

(i) By considering the image of a particular point under M,

explain why $a^2 + b^2 = 1$.

(ii) Given that a necessary and sufficient condition for a

transformation to have a line of invariant points is that

$tr(M) = |M| + 1$ (where $tr(M)$ (the trace of M) $= a + d$)

and by considering the image of a point on the line $y = mx$ (or

otherwise), show that $M = \begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{pmatrix}$