

Matrices : General - Exercises (with Sol'ns)

(7 pages ; 3/6/26)

(1) Find k such that $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and $\begin{pmatrix} 5 & 9 \\ 6 & k \end{pmatrix}$ commute.

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Solution

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 5 & 9 \\ 6 & k \end{pmatrix} = \begin{pmatrix} 23 & 9 + 3k \\ 34 & 18 + 4k \end{pmatrix}$$

$$\begin{pmatrix} 5 & 9 \\ 6 & k \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 23 & 51 \\ 6 + 2k & 18 + 4k \end{pmatrix}$$

Then $34 = 6 + 2k$, so that $k = 14$

And $9 + 3k = 51$, so that $k = 14$ also.

(2) Suppose that the following pair of equations enables (x', t') to be determined from (x, t) :

$$x' = \gamma(x - vt) \text{ \& } t' = \gamma(t - \frac{xv}{c^2}) \quad (\text{A})$$

and that it is also true that

$$x = \gamma(x' + vt') \text{ \& } t = \gamma(t' + \frac{x'v}{c^2}) \quad (\text{B})$$

[These are the transformation equations in Special Relativity between two frames of reference that are moving with a relative speed of v . Starting with (A), (B) is obtained by reversing the roles of the two frames (so that the speed is reversed as well).]

Use matrix multiplication to find an expression for γ in terms of v & c .

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Solution

$$x' = \gamma(x - vt) \text{ \& } t' = \gamma(t - \frac{xv}{c^2})$$

$$\Rightarrow \begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

$$\text{and } x = \gamma(x' + vt') \text{ \& } t = \gamma(t' + \frac{x'v}{c^2})$$

$$\Rightarrow \begin{pmatrix} x \\ t \end{pmatrix} = \gamma \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$$

$$\text{Hence } \begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$$

$$\text{and so } \gamma \begin{pmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{giving } \gamma^2 \begin{pmatrix} 1 - \frac{v^2}{c^2} & 0 \\ 0 & 1 - \frac{v^2}{c^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{and hence } \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

[This is the Lorentz factor.]

(3) Find the condition(s) for two 2×2 matrices to commute.

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Solution

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} e & g \\ f & h \end{pmatrix} = \begin{pmatrix} e & g \\ f & h \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\Rightarrow ae + cf = ae + bg \Rightarrow \frac{b}{c} = \frac{f}{g} \quad (1)$$

Also $bg + dh = cf + dh \Rightarrow$ same condition

Then $be + df = af + bh$ (2) and $ag + ch = ce + dg$ (3)

(2) $\Rightarrow b(e - h) = f(a - d)$ and (3) $\Rightarrow c(h - e) = g(d - a)$

From (1), $\frac{b}{f} = \frac{c}{g}$ and so both of the above produce the same condition:

$$\frac{b}{f} = \frac{a-d}{e-h} \Rightarrow \frac{a-d}{b} = \frac{e-h}{f} \quad (4)$$

Thus, two 2×2 matrices commute if the quantities $\frac{b}{c}$ and $\frac{a-d}{b}$ in one matrix match the corresponding quantities in the other.

As an example, we could choose the matrices $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and $\begin{pmatrix} 5 & g \\ 6 & h \end{pmatrix}$.

Then $g = 6 \times \frac{3}{2} = 9$ and $\frac{h-5}{6} = \frac{4-1}{2} \Rightarrow h = 14$

$$\text{Check: } \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 5 & 9 \\ 6 & 14 \end{pmatrix} = \begin{pmatrix} 23 & 51 \\ 34 & 74 \end{pmatrix}$$

$$\text{and } \begin{pmatrix} 5 & 9 \\ 6 & 14 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 23 & 51 \\ 34 & 74 \end{pmatrix}$$

To test the conditions on a matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and its inverse,

$$\frac{1}{ad-bc} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$$(i) \frac{-b/(ad-bc)}{-c/(ad-bc)} = \frac{b}{c}$$

$$(ii) \frac{(d-a)/(ad-bc)}{-b/(ad-bc)} = \frac{a-d}{b}$$