Matrices - Exercises: Transformations (4 pages; 22/2/20)

Key to difficulty:

* easier

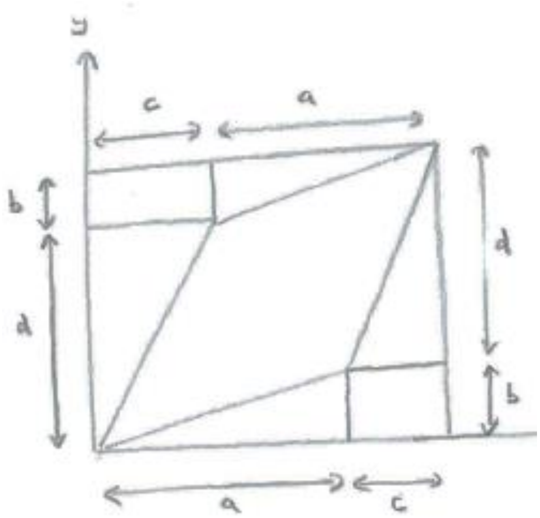
** moderate

*** harder

(1*) How can you tell if a matrix represents a (pure) reflection / rotation?

(2*) Derive a formula for the area of a triangle with corners at (0,0), (a, b), (c, d) , using matrix transformations.

(3*) Use the diagram below to show that the area scale factor of the transformation represented by $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is the determinant of the matrix.



(4***) Find the equation of the line that the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ maps all points to, and find the equation of the line that maps to the point (1,3)

(5***) Find the equations of the invariant lines of the transformation represented by the matrix $\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}$

(6***) Prove that invariant points of a 2D transformation always lie on a line passing through the Origin.

(7***) In 3 dimensions, find the effect of a reflection in the plane $y = 0$, followed by a reflection in the plane $x = 0$

(8***) Show that the matrix $\begin{pmatrix} cos2\theta & sin2\theta \\ sin2\theta & -cos2\theta \end{pmatrix}$ [representing a reflection in the line $y = tan\theta . x$] can be written as $\begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{pmatrix}$

- where $m = tan\theta$

(9***) For the transformation matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$, where a, b, c & d are positive, find a relationship between the trace $a + d$ and the determinant that must hold in order for the transformation to have an invariant line that doesn't pass through the Origin.

(10***) For the matrix $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$, find all of the invariant lines of the associated transformation.

(11***) If the condition found in Exercise (8) applies, what can be deduced about the eigenvalues of the transformation?

(12***) For a reflection in the line $y = x$:

(i) Find the transformation matrix.

(ii) Use eigenvectors to find the invariant lines through the Origin.

(iii) By drawing a diagram, are there any invariant lines that don't pass through the Origin?

(iv) Are there any lines of invariant points?

(13***) Consider the transformation represented by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

(i) What type of transformation is this?

(ii) Use eigenvectors to find the invariant lines through the Origin.

(iii) What can be said about the line $y = 0$?

(iv) What can be said about the line $y = c$? , where $c \neq 0$

(14***) Consider the transformation represented by the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$

(i) Write down the eigenvalues, without forming the characteristic equation.

(ii) Find the invariant lines passing through the Origin.

(iii) Show that all points in the plane are transformed to points on a specific line (to be found).

(iv) Find the line whose points are transformed to (1,3).

(v) State the line whose points are transformed to the Origin.

(15***) Find the invariant points and lines for the transformation represented by

the matrix $\begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}$.

(16***)(i) Show that the transformation represented by the matrix $\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$ (with determinant zero) maps all points to a particular line.

(ii) Find the line whose points all map to the point $(3,1)$.

(iii) Without doing any calculations, what can be said about the line whose points all map to the point $(6,2)$?

(iv) Write down the line whose points all map to the Origin.