Matrices - Exercises: Simultaneous Eq'ns (Solutions)

(8 pages; 31/3/20)

(1**)(i) Three planes are represented by the following equations:

$x - y + z = 1$

$2x + ky + 2z = 3$

$x + 3y + 3z = 5$

For what value of k do the planes not meet at a single point? For this value of k how are the planes configured?

(ii) If $k = 2$, find the point of intersection, using matrices.

Solution

(i) $\begin{pmatrix} 1 & -1 & 1 \\ 2 & k & 2 \\ 1 & 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$

$\begin{vmatrix} 1 & -1 & 1 \\ 2 & k & 2 \\ 1 & 3 & 3 \end{vmatrix} = (3k - 6) + (6 - 2) + (6 - k)$ [expanding by the 1st row]

$= 2k + 4$

The equations don't have a unique solution when $2k + 4 = 0$; ie $k = -2$

In that case, the equations are:

$x - y + z = 1$

$2x - 2y + 2z = 3$

$x + 3y + 3z = 5$

As the direction vectors of the first two planes are $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}2\\-2\\2\end{pmatrix}$, which are equivalent, and the constant terms on the RHS are not in the same ratio as the LHS terms, these planes are parallel, and the 3rd plane cuts both of the other planes (not being parallel to either of them).

(ii) To solve $\begin{pmatrix}1&-1&1\\2&2&2\\1&3&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\3\\5\end{pmatrix}$: det $=2k+4=8$

and so $\begin{pmatrix}1&-1&1\\2&2&2\\1&3&3\end{pmatrix}^{-1}=\frac{1}{8}\begin{pmatrix}0&-4&4\\6&2&-4\\-4&0&4\end{pmatrix}^{T}=$
$\frac{1}{8}\begin{pmatrix}0&6&-4\\-4&2&0\\4&-4&4\end{pmatrix}$

[eg $6=-\left((-1)\times 3-3\times 1\right)$; $2=1\times 3-1\times 1$;
$-4=-\left(1\times 3-1\times(-1)\right)$]

So $\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{8}\begin{pmatrix}0&6&-4\\-4&2&0\\4&-4&4\end{pmatrix}\begin{pmatrix}1\\3\\5\end{pmatrix}=\frac{1}{8}\begin{pmatrix}-2\\2\\12\end{pmatrix}=\frac{1}{4}\begin{pmatrix}-1\\1\\6\end{pmatrix}$

ie $x=\frac{-1}{4}$, $y=\frac{1}{4}$, $z=\frac{3}{2}$

(2**) Find the value of k for which the following equations are consistent.

$3x - 3y - z = k$

$2x - y - z = 5$

$x + 4y - 2z = 7$

Solution

$3x - 3y - z = k$ (1)

$2x - y - z = 5$ (2)

$x + 4y - 2z = 7$ (3)

Method 1

Using (2) to eliminate z in (1) & (3):

$3x - 3y - (2x - y - 5) = k$; ie $x - 2y = k - 5$ (1')

$x + 4y - 2(2x - y - 5) = 7$; ie $-3x + 6y = -3$

and $x - 2y = 1$ (3')

Hence, $k - 5 = 1$ for consistency, so that $k = 6$

Method 2

$$\begin{vmatrix} 3 & -3 & -1 \\ 2 & -1 & -1 \\ 1 & 4 & -2 \end{vmatrix} = 3(6) - 2(10) + 1(2) = 0$$

By Cramer's rule, $x = \dfrac{\begin{vmatrix} k & -3 & -1 \\ 5 & -1 & -1 \\ 7 & 4 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -3 & -1 \\ 2 & -1 & -1 \\ 1 & 4 & -2 \end{vmatrix}}$, and this will only have a value if

$$\begin{vmatrix} k & -3 & -1 \\ 5 & -1 & -1 \\ 7 & 4 & -2 \end{vmatrix} = 0$$

ie when $k(6) - 5(10) + 7(2) = 0$,

so that $6k = 36$; $k = 6$

(3**) Show that the following three planes meet in a line, giving the equation of that line in cartesian form.

$x - y + 3z = 4$
$4x + 5y - 2z = 8$
$x + 17y - 25z = -12$

Solution

First of all, none of the lines are parallel to each other.

Then $\begin{vmatrix} 1 & -1 & 3 \\ 4 & 5 & -2 \\ 1 & 17 & -25 \end{vmatrix} = 1(-91) - (-1)(-98) + 3(63) = 0$

[as expected for this sort of question]

So the planes will either be configured as a sheaf (if they have a line of intersection) or as a triangular prism (if not).

[In some cases it may be possible to spot that one equation is a combination of the other two, showing that the equations are consistent, and that they meet in a line.]

$x - y + 3z = 4 \quad (1)$
$4x + 5y - 2z = 8 \; (2)$
$x + 17y - 25z = -12(3)$

Substituting for x (say), from (1) into (2) gives:

$4(4 + y - 3z) + 5y - 2z = 8$, so that $9y - 14z = -8$

Substituting into (3) gives:

$(4 + y - 3z) + 17y - 25z = -12$, so that $18y - 28z = -16$,

which is the same equation, and hence the planes meet as a sheaf.

To find the line of intersection, let $x = \lambda$ (say).

Then, from (1), $-y + 3z = 4 - \lambda$ (3)

and from (2), $5y - 2z = 8 - 4\lambda$ (4)

Then $5(3) + (4) \Rightarrow 13z = 28 - 9\lambda$

and $2(3) + 3(4) \Rightarrow 13y = 32 - 14\lambda$,

so that the equation of the line is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 13\lambda \\ 32 - 14\lambda \\ 28 - 9\lambda \end{pmatrix}$

or $\frac{x}{13} = \frac{y - \frac{32}{13}}{-14} = \frac{z - \frac{28}{13}}{-9}$

[As a check, points on the line where $\lambda = 0$ and 1 could be substituted into the equations of the planes.

Also, it can be shown that the determinant formed by replacing (any) one of the columns of the matrix by the right-hand values will be zero when the equations are consistent. (Consider the 2×2 case to see why this is likely to be true.)

Thus $\begin{vmatrix} 1 & -1 & 4 \\ 4 & 5 & 8 \\ 1 & 17 & -12 \end{vmatrix} = 1(-196) - (-1)(-56) + 4(63) = 0$, for example.]

(4**) Consider the planes with the following equations:

$$\begin{aligned} ax - y + z &= 1 \\ 2y - z &= b \\ 4x + 3y - 2z &= 2 \end{aligned}$$

(i) Find conditions on a and b for:

(a) the 3 planes to meet at a single point
(b) the 3 planes to meet in a line
(c) no point of intersection of the 3 planes

(ii) Show that in case (c) the line of intersection of the 1st two planes is parallel to the 3rd plane.

Solution

(a) $\Delta = \begin{vmatrix} a & -1 & 1 \\ 0 & 2 & -1 \\ 4 & 3 & -2 \end{vmatrix}$ = (expanding by the 1st column)

$a(-1) + 4(-1)$

The 3 planes will meet at a single point when $\Delta \neq 0$; ie when $a \neq -4$

(b) The 3 planes will meet in a line (as a sheaf of planes) when $\Delta = 0$ (ie $a = -4$) and the equations are consistent.

Method 1

$$\begin{aligned} -4x - y + z &= 1 \quad (1) \\ 2y - z &= b \quad (2) \\ 4x + 3y - 2z &= 2 \quad (3) \end{aligned}$$

(1) + (3) gives $2y - z = 3$

This is consistent with (2) when $b = 3$

Method 2

Replacing eg the 3rd column of the determinant with $\begin{pmatrix} 1 \\ b \\ 2 \end{pmatrix}$,

the equations will be consistent when $\begin{vmatrix} -4 & -1 & 1 \\ 0 & 2 & b \\ 4 & 3 & 2 \end{vmatrix} = 0$

(this is a result connected with Cramer's method for solving simultaneous equations)
Expanding about the 2nd row,

$$\begin{vmatrix} -4 & -1 & 1 \\ 0 & 2 & b \\ 4 & 3 & 2 \end{vmatrix} = 0 \Rightarrow 2(-12) - b(-8) = 0 \Rightarrow b = 3$$

Method 3

We can attempt to find a common line:

Let (eg) $x = \lambda$, so that

$$-y + z = 1 + 4\lambda \quad (1)$$
$$2y - z = b \quad (2)$$
$$3y - 2z = 2 - 4\lambda \quad (3)$$

Then (1) + (2) gives $y = 1 + b + 4\lambda$

and (2) + 2(1) gives $z = b + 2 + 8\lambda$

Substituting into (3): $3(1 + b + 4\lambda) - 2(b + 2 + 8\lambda) = 2 - 4\lambda$

$\Rightarrow -3 + b = 0 \Rightarrow b = 3$

(c) There will be no point of intersection of the 3 planes (which will then form a triangular prism) when $a = -4$ and $b \neq 3$

(ii) To find the intersection of the first two planes:

$$-4x - y + z = 1$$
$$2y - z = b$$

Let eg $x = \lambda$, so that:

$-y+z=1+4\lambda$ (1)
$2y-z=b$ (2)

(1) + (2) gives $y=1+4\lambda+b$

And 2(1) + (2) gives $z=2+8\lambda+b$

Hence the line of intersection of the first two planes is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1+b \\ 2+b \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix}$$

Then $\begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} . \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} = 4+12-16=0$,

so that this line is perpendicular to the normal vector to the 3rd plane, and hence parallel to the plane.