Matrices - Exercises: General (Solutions)

(9 pages; 31/3/20)

(1**) Prove that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

Solution

Suppose that $\begin{pmatrix} e & g \\ f & h \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Then $af + bh = 0$ & $ce + dg = 0$

So $h = -\frac{af}{b}$ & $g = -\frac{ce}{d}$ (*)

Also $ae + bg = 1$ & $cf + dh = 1$,

so that $ae - \frac{bce}{d} = 1 \Rightarrow e(ad - bc) = d$

and $cf - \frac{daf}{b} = 1 \Rightarrow f(bc - ad) = b$

Let $\Delta = ad - bc$

Then $e = \frac{d}{\Delta}$ & $f = -\frac{b}{\Delta}$

And, from (*), $g = -\frac{c}{\Delta}$ & $h = \frac{a}{\Delta}$

Thus $\begin{pmatrix} e & g \\ f & h \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

(2***) Show that if N is the left inverse of M, so that $NM = I$, then it is also the right inverse.

Solution

Define N^L to be the left inverse of N, so that $N^L N = I$

$NM = I$

$\Rightarrow N^L(NM) = N^L I = N^L$

$\Rightarrow (N^L N)M = N^L$

$\Rightarrow IM = N^L$

$\Rightarrow M = N^L$

$\Rightarrow MN = N^L N = I$

ie N is the right inverse of M

(3**) Prove that $(AB)^{-1} = B^{-1}A^{-1}$

Solution

Let $X = AB$

Then $XX^{-1} = I$, so that $ABX^{-1} = I$

Hence $A^{-1}ABX^{-1} = A^{-1}I$,

so that $BX^{-1} = A^{-1}$

Then $B^{-1}BX^{-1} = B^{-1}A^{-1}$,

so that $X^{-1} = B^{-1}A^{-1}$

(4***) Suppose that the following pair of equations enables (x', t') to be determined from (x, t):

$x' = \gamma(x - vt)$ & $t' = \gamma(t - \frac{xv}{c^2})$ (A)

and that it is also true that

$x = \gamma(x' + vt')$ & $t = \gamma(t' + \frac{x'v}{c^2})$ (B)

[These are the transformation equations in Special Relativity between two frames of reference that are moving with a relative speed of v. Starting with (A), (B) is obtained by reversing the roles of the two frames (so that the speed is reversed as well).]

Use matrix multiplication to find an expression for γ in terms of v & c.

Solution

$x' = \gamma(x - vt)$ & $t' = \gamma(t - \frac{xv}{c^2})$

$\Rightarrow \begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$

and $x = \gamma(x' + vt')$ & $t = \gamma(t' + \frac{x'v}{c^2})$

$\Rightarrow \begin{pmatrix} x \\ t \end{pmatrix} = \gamma \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$

Hence $\begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$

and so $\gamma \begin{pmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

giving $\gamma^2 \begin{pmatrix} 1 - \frac{v^2}{c^2} & 0 \\ 0 & 1 - \frac{v^2}{c^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and hence $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

[This is the Lorentz factor.]

(5***) Assuming that $(AB)^T = B^T A^T$, prove that $(A^T)^{-1} = (A^{-1})^T$

Solution

Let $B = (A^T)^{-1}$, so that $BA^T = I$ (1)

Result to prove: $B = (A^{-1})^T$

[Noting that this is equivalent to $B^T = A^{-1}$, it seems promising to involve B^T]

From (1), $(BA^T)^T = I^T = I$, so that $AB^T = I$,

and hence $B^T = A^{-1}$ and $B = (A^{-1})^T$, as required.

(6***) Factorise the determinant $\begin{vmatrix} x^2-x & y^2-y & z^2-z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$

Solution

$C2 \to C2 - C1 \,\&\, C3 \to C3 - C1 \Rightarrow$

$$\begin{vmatrix} x^2-x & y^2-y-x^2+x & z^2-z-x^2+x \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} x^2-x & (y^2-x^2)-(y-x) & (z^2-x^2)-(z-x) \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} x^2-x & (y-x)(y+x-1) & (z-x)(z+x-1) \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} x^2-x & y+x-1 & z+x-1 \\ x & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (y-x)(z-x)\{y+x-1-(z+x-1)\}$$

$$= (y-x)(z-x)(y-z)$$

Alternatively:

$R1 \to R1 + R2 \Rightarrow \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$

$C2 \to C2 - C1 \,\&\, C3 \to C3 - C1 \Rightarrow \begin{vmatrix} x^2 & y^2-x^2 & z^2-x^2 \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$

$$= (y-x)(z-x)\begin{vmatrix} x^2 & y+x & z+x \\ x & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (y-x)(z-x)(y-z)$$

(7***) Write the determinant $\begin{vmatrix} \mathbf{1} & \boldsymbol{x^2} & \boldsymbol{x^4} \\ \mathbf{1} & \boldsymbol{y^2} & \boldsymbol{y^4} \\ \mathbf{1} & \boldsymbol{z^2} & \boldsymbol{z^4} \end{vmatrix}$ as a product of linear factors.

Solution

Replacing row 1 with row 1 - row 2,

$$\boldsymbol{D} = \begin{vmatrix} \mathbf{1} & \boldsymbol{x^2} & \boldsymbol{x^4} \\ \mathbf{1} & \boldsymbol{y^2} & \boldsymbol{y^4} \\ \mathbf{1} & \boldsymbol{z^2} & \boldsymbol{z^4} \end{vmatrix} = \begin{vmatrix} \mathbf{0} & \boldsymbol{x^2 - y^2} & \boldsymbol{x^4 - y^4} \\ \mathbf{1} & \boldsymbol{y^2} & \boldsymbol{y^4} \\ \mathbf{1} & \boldsymbol{z^2} & \boldsymbol{z^4} \end{vmatrix}$$

$$= (\boldsymbol{x^2 - y^2}) \begin{vmatrix} \mathbf{0} & \mathbf{1} & \boldsymbol{x^2 + y^2} \\ \mathbf{1} & \boldsymbol{y^2} & \boldsymbol{y^4} \\ \mathbf{1} & \boldsymbol{z^2} & \boldsymbol{z^4} \end{vmatrix}$$

Similarly, replacing row 2 with row 2 - row 3,

$$\boldsymbol{D} = (\boldsymbol{x^2 - y^2})(\boldsymbol{y^2 - z^2}) \begin{vmatrix} \mathbf{0} & \mathbf{1} & \boldsymbol{x^2 + y^2} \\ \mathbf{0} & \mathbf{1} & \boldsymbol{y^2 + z^2} \\ \mathbf{1} & \boldsymbol{z^2} & \boldsymbol{z^4} \end{vmatrix}$$

$$= (\boldsymbol{x^2 - y^2})(\boldsymbol{y^2 - z^2})(\boldsymbol{y^2 + z^2} - [\boldsymbol{x^2 + y^2}])$$

$$= (\boldsymbol{x^2 - y^2})(\boldsymbol{y^2 - z^2})(\boldsymbol{z^2 - x^2})$$

$$= (\boldsymbol{x - y})(\boldsymbol{x + y})(\boldsymbol{y - z})(\boldsymbol{y + z})(\boldsymbol{z - x})(\boldsymbol{z + x})$$

(8***) Find the condition(s) for two 2×2 matrices to commute.

Solution

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} e & g \\ f & h \end{pmatrix} = \begin{pmatrix} e & g \\ f & h \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$\Rightarrow ae + cf = ae + bg \Rightarrow \frac{b}{c} = \frac{f}{g}$ (1)

Also $bg + dh = cf + dh \Rightarrow$ same condition

Then $be + df = af + bh$ (2) and $ag + ch = ce + dg$ (3)

(2) $\Rightarrow b(e-h) = f(a-d)$ and (3) $\Rightarrow c(h-e) = g(d-a)$

From (1), $\frac{b}{f} = \frac{c}{g}$ and so both of the above produce the same condition:

$\frac{b}{f} = \frac{a-d}{e-h} \Rightarrow \frac{a-d}{b} = \frac{e-h}{f}$ (4)

Thus, two 2×2 matrices commute if the quantities $\frac{b}{c}$ and $\frac{a-d}{b}$ in one matrix match the corresponding quantities in the other.

As an example, we could choose the matrices $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and $\begin{pmatrix} 5 & g \\ 6 & h \end{pmatrix}$.

Then $g = 6 \times \frac{3}{2} = 9$ and $\frac{h-5}{6} = \frac{4-1}{2} \Rightarrow h = 14$

Check: $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 5 & 9 \\ 6 & 14 \end{pmatrix} = \begin{pmatrix} 23 & 51 \\ 34 & 74 \end{pmatrix}$

and $\begin{pmatrix} 5 & 9 \\ 6 & 14 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 23 & 51 \\ 34 & 74 \end{pmatrix}$

To test the conditions on a matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and its inverse,

$\frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

(i) $\frac{-b/(ad-bc)}{-c/(ad-bc)} = \frac{b}{c}$

(ii) $\frac{(d-a)/(ad-bc)}{-b/(ad-bc)} = \frac{a-d}{b}$

(9***) Given that a 3×3 determinant can always be reduced to triangular form (in the same way as simultaneous equations), to produce a multiple of $\begin{vmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix}$, show that it can be further reduced to a multiple of the Identity matrix. [Obviously this is an academic exercise, as the determinant can be evaluated as soon as triangular form has been reached.]

Solution

From $\begin{vmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix}$, if $R1 \to R1 - a(R2)$, we get $\begin{vmatrix} 1 & 0 & b-ac \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix}$

Then, $C3 \to C3 - c(C2)$ gives $\begin{vmatrix} 1 & 0 & b-ac \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

and finally $R1 \to R1 - (b-ac)(R3)$ gives $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

[Note that no further factors have had to be taken outside the determinant - as expected, since $\begin{vmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix} = 1$]

(10***) Show that a matrix is orthogonal if and only if

(i) its columns are mutually orthogonal (ie perpendicular, so that their scalar product is zero), and
(ii) each column has unit magnitude

Solution

A matrix P is orthogonal when $P^{-1} = P^T$; ie when $PP^T = I$

Suppose that $P = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$,

so that $PP^T = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

Then the diagonal entries will be 1 when the 3 columns of P have unit magnitude, and the non-diagonal entries will be 0 when the columns are mutually orthogonal.

(11***) Find c, a & b such that $\begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix} = a\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + b\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$

[ie such that the 3 vectors are not linearly independent]

Solution

As the position vector $\begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix}$ is in the plane containing the Origin and the position vectors $\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$, it follows that $\begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix}$ is perpendicular to the normal to that plane; ie perpendicular to

$\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} = \begin{vmatrix} \underline{i} & -1 & 0 \\ \underline{j} & 0 & 2 \\ \underline{k} & 3 & 4 \end{vmatrix}$; so that

$\begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix} . \begin{vmatrix} \underline{i} & -1 & 0 \\ \underline{j} & 0 & 2 \\ \underline{k} & 3 & 4 \end{vmatrix} = 0$, and thus $\begin{vmatrix} 2 & -1 & 0 \\ 3 & 0 & 2 \\ c & 3 & 4 \end{vmatrix} = 0$

Alternative Approach 1

The 3 vectors form a parallelepiped of zero volume, so that the scalar triple product of the vectors is zero.]

Alternative Approach 2

The required relation can be written as

$$\begin{pmatrix}2\\3\\c\end{pmatrix} - a\begin{pmatrix}-1\\0\\3\end{pmatrix} - b\begin{pmatrix}0\\2\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix},$$

which implies a solution of $x\begin{pmatrix}2\\3\\c\end{pmatrix} + y\begin{pmatrix}-1\\0\\3\end{pmatrix} + z\begin{pmatrix}0\\2\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

other than $x = y = z = 0$,

and for there to be more than one solution to this matrix equation, we require the determinant to be zero. ■

Then $\begin{vmatrix}2 & -1 & 0\\3 & 0 & 2\\c & 3 & 4\end{vmatrix} = 0 \Rightarrow 2(-6) - (-1)(12 - 2c) = 0$

$\Rightarrow c = 0$

$$\begin{pmatrix}2\\3\\0\end{pmatrix} = a\begin{pmatrix}-1\\0\\3\end{pmatrix} + b\begin{pmatrix}0\\2\\4\end{pmatrix}$$

$\Rightarrow 2 = -a$

$3 = 2b$

& $0 = 3a + 4b$

so that $a = -2$ & $b = \frac{3}{2}$