Matrices - Exercises: General (3 pages; 31/3/20)

Key to difficulty:

* easier

** moderate

*** harder

(1**) Prove that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

(2***) Show that if N is the left inverse of M, so that $NM = I$, then it is also the right inverse.

(3**) Prove that $(AB)^{-1} = B^{-1}A^{-1}$

(4***) Suppose that the following pair of equations enables (x', t') to be determined from (x, t):

$x' = \gamma(x - vt)$ & $t' = \gamma(t - \frac{xv}{c^2})$ (A)

and that it is also true that

$x = \gamma(x' + vt')$ & $t = \gamma(t' + \frac{x'v}{c^2})$ (B)

[These are the transformation equations in Special Relativity between two frames of reference that are moving with a relative speed of v. Starting with (A), (B) is obtained by reversing the roles of the two frames (so that the speed is reversed as well).]

Use matrix multiplication to find an expression for γ in terms of v & c.

(5***) Assuming that $(AB)^T = B^T A^T$, prove that $(A^T)^{-1} = (A^{-1})^T$

(6***) Factorise the determinant $\begin{vmatrix} x^2-x & y^2-y & z^2-z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$

(7***) Write the determinant $\begin{vmatrix} \mathbf{1} & \boldsymbol{x^2} & \boldsymbol{x^4} \\ \mathbf{1} & \boldsymbol{y^2} & \boldsymbol{y^4} \\ \mathbf{1} & \boldsymbol{z^2} & \boldsymbol{z^4} \end{vmatrix}$ as a product of linear factors.

(8***) Find the condition(s) for two 2×2 matrices to commute.

(9***) Given that a 3×3 determinant can always be reduced to triangular form (in the same way as simultaneous equations), to produce a multiple of $\begin{vmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix}$, show that it can be further reduced to a multiple of the Identity matrix. [Obviously this is an academic exercise, as the determinant can be evaluated as soon as triangular form has been reached.]

(10***) Show that a matrix is orthogonal if and only if

(i) its columns are mutually orthogonal (ie perpendicular, so that their scalar product is zero), and
(ii) each column has unit magnitude

(11***) Find c, a & b such that $\begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix} = a\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + b\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$

[ie such that the 3 vectors are not linearly independent]