Matrices – Determinants: Exercises (with Sol'ns)

(8 pages ; 3/6/26)

(1) If $M = \begin{pmatrix} \lambda & k \\ 1 & \lambda - k \end{pmatrix}$, where λ & k are real numbers, what is the range of possible values of k, in order that $|M| > 0$ for all values of λ?

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Solution

$\begin{vmatrix} \lambda & k \\ 1 & \lambda - k \end{vmatrix} > 0 \Rightarrow \lambda(\lambda - k) - k > 0$

$\Rightarrow f(\lambda) = \lambda^2 - k\lambda - k > 0$

This means that the (u-shaped) quadratic curve $y = f(x)$ lies entirely above the x-axis.

This will occur when the discriminant, $(-k)^2 - 4(-k) < 0$

ie $k(k + 4) < 0$

$\Rightarrow -4 < k < 0$

(2) Factorise the determinant $\begin{vmatrix} x^2 - x & y^2 - y & z^2 - z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$

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Solution

$C2 \to C2 - C1$ & $C3 \to C3 - C1 \Rightarrow$

$$\begin{vmatrix} x^2-x & y^2-y-x^2+x & z^2-z-x^2+x \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} x^2-x & (y^2-x^2)-(y-x) & (z^2-x^2)-(z-x) \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} x^2-x & (y-x)(y+x-1) & (z-x)(z+x-1) \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} x^2-x & y+x-1 & z+x-1 \\ x & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (y-x)(z-x)\{y+x-1-(z+x-1)\}$$

$$= (y-x)(z-x)(y-z)$$

Alternatively:

$R1 \to R1 + R2 \Rightarrow \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$

$C2 \to C2 - C1$ & $C3 \to C3 - C1 \Rightarrow \begin{vmatrix} x^2 & y^2-x^2 & z^2-x^2 \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$

$$= (y-x)(z-x)\begin{vmatrix} x^2 & y+x & z+x \\ x & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (y-x)(z-x)(y-z)$$

(3) Write the determinant $\begin{vmatrix} 1 & x^2 & x^4 \\ 1 & y^2 & y^4 \\ 1 & z^2 & z^4 \end{vmatrix}$ as a product of linear factors.

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Solution

Replacing row 1 with row 1 - row 2,

$$D = \begin{vmatrix} 1 & x^2 & x^4 \\ 1 & y^2 & y^4 \\ 1 & z^2 & z^4 \end{vmatrix} = \begin{vmatrix} 0 & x^2 - y^2 & x^4 - y^4 \\ 1 & y^2 & y^4 \\ 1 & z^2 & z^4 \end{vmatrix}$$

$$= (x^2 - y^2) \begin{vmatrix} 0 & 1 & x^2 + y^2 \\ 1 & y^2 & y^4 \\ 1 & z^2 & z^4 \end{vmatrix}$$

Similarly, replacing row 2 with row 2 - row 3,

$$D = (x^2 - y^2)(y^2 - z^2) \begin{vmatrix} 0 & 1 & x^2 + y^2 \\ 0 & 1 & y^2 + z^2 \\ 1 & z^2 & z^4 \end{vmatrix}$$

$$= (x^2 - y^2)(y^2 - z^2)(y^2 + z^2 - [x^2 + y^2])$$

$$= (x^2 - y^2)(y^2 - z^2)(z^2 - x^2)$$

$$= (x - y)(x + y)(y - z)(y + z)(z - x)(z + x)$$

(4) Given that a 3×3 determinant can always be reduced to triangular form (in the same way as simultaneous equations), to produce a multiple of $\begin{vmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix}$, show that it can be further reduced to a multiple of the Identity matrix. [Obviously this is an academic exercise, as the determinant can be evaluated as soon as triangular form has been reached.]

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Solution

From $\begin{vmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix}$, if $R1 \rightarrow R1 - a(R2)$, we get $\begin{vmatrix} 1 & 0 & b - ac \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix}$

Then, $C3 \rightarrow C3 - c(C2)$ gives $\begin{vmatrix} 1 & 0 & b - ac \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

and finally $R1 \rightarrow R1 - (b - ac)(R3)$ gives $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

[Note that no further factors have had to be taken outside the determinant - as expected, since $\begin{vmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix} = 1$]