Maclaurin Series - Exercises (Solutions) (8 pages; 22/3/20)

(1***) Find the 1st 3 non-zero terms of the Maclaurin expansions of the following functions, and the intervals of validity of the infinite series:

(i) $\ln(3-2x)$

(ii) $ln\left(\frac{\sqrt{1+3x}}{1-2x}\right)$

(iii) e^{cosx}

Solution

(i) $\ln(3-2x) = ln3\left(1-\frac{2x}{3}\right) = ln3 + \ln(1-\frac{2x}{3})$

$= ln3 + \left(-\frac{2x}{3}\right) - \frac{\left(-\frac{2x}{3}\right)^2}{2} + \cdots \approx ln3 - \frac{2x}{3} - \frac{2x^2}{9}$

The infinite series is valid for $-1 < -\frac{2x}{3} \leq 1$; ie $-\frac{3}{2} \leq x < \frac{3}{2}$

[Note: Were we to go down the route of

$\ln(3-2x) = \ln(1+[2-2x])$, there would be an infinite number of terms involving a constant, arising from the powers of $2-2x$]

(ii) $ln\left(\frac{\sqrt{1+3x}}{1-2x}\right) = \frac{1}{2}\ln(1+3x) - \ln(1-2x)$

$= \frac{1}{2}(3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} + \cdots)$

$-([-2x] - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} - \frac{(-2x)^4}{4} + \cdots)$

$= \frac{7x}{2} + x^2\left(-\frac{9}{4} + 2\right) + x^3\left(\frac{9}{2} + \frac{8}{3}\right) + \cdots$

$= \frac{7x}{2} - \frac{x^2}{4} + \frac{43x^3}{6} + \cdots$

valid for x such that $-1 < 3x \leq 1$ and $-1 < -2x \leq 1$

ie $-\frac{1}{3} < x \leq \frac{1}{3}$ and $\frac{1}{2} > x \geq -\frac{1}{2}$

ie $-\frac{1}{3} < x \leq \frac{1}{3}$

(iii) $e^{cosx} = e^{1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots}$

$= e\{1 + \left[-\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right] + \frac{1}{2}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2 + \cdots\}$

$= e\{1 - \frac{x^2}{2} + x^4\left(\frac{1}{24} + \frac{1}{8}\right) + \cdots\}$

$= e - \frac{ex^2}{2} + \frac{ex^4}{6} + \cdots$

valid for all x (as both $cosx$ & e^x are valid for all values).

[Note that the expansion $e^{1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots}$

$= 1 + \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right] + \frac{1}{2}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2 + \cdots$ would have involved an infinite number of constant terms (though their sum would been e).]

(2*) Find a Maclaurin expansion (with 3 non-zero terms) for sin^2x by two methods

Solution

Method 1

$sin^2x = (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)$

$= x^2 + \frac{x^6}{(3!)^2} + 2(x)\left(-\frac{x^3}{3!}\right) + 2x\left(\frac{x^5}{5!}\right) + \cdots$

$= x^2 - \frac{x^4}{3} + \left(\frac{1}{36} + \frac{1}{60}\right)x^6 + \cdots$

[though in fact the next term is negative]

$= x^2 - \frac{x^4}{3} + \frac{8}{180}x^6 + \cdots = x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 + \cdots$

Method 2

$sin^2x = \frac{1}{2}(1 - cos2x)$

$= \frac{1}{2} - \frac{1}{2}(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots)$

$x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 - \cdots$

(3*) Expand $\sqrt{x}$ in powers of $x - 1$, and investigate the accuracy of the resulting approximation for $\sqrt{2}$ when 8 terms are taken.

Solution

$\sqrt{x} = (1 + [x-1])^{\frac{1}{2}} = 1 + \frac{1}{2}(x-1) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(x-1)^2$

$+ \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(x-1)^3 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{4!}(x-1)^4 + \cdots$

Hence $\sqrt{2} \approx 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{5}{128} + \frac{7}{256} - \frac{21}{1024} + \frac{33}{2048} = 1.42139$

(true figure: 1.41421)

(4***) Find the % errors (to 2sf) associated with the following Maclaurin approximations:

(i)(a) $\cos x = 1 - \frac{x^2}{2!}$ for $x = \frac{\pi}{6}$ & $x = \frac{\pi}{3}$

(b) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ for $x = \frac{\pi}{6}$ & $x = \frac{\pi}{3}$

(ii)(a) $e^x = 1 + x + \frac{x^2}{2!}$ for $x = 1$ & $x = 2$

(b) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$ for $x = 1$ & $x = 2$

(iii) $\ln(1+x) = x - \frac{1}{2}x^2$ for $x = 0.1, x = 0.5$ & $x = 1$

(b) $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$

for $x = 0.1$, $x = 0.5$ & $x = 1$

Solution

(i)(a) $cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} = 0.86603$; $1 - \frac{\left(\frac{\pi}{6}\right)^2}{2!} = 0.86292$

% error: $\left|\frac{0.86292-0.86603}{0.86603}\right| \times 100 = 0.36\%$

$cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$; $1 - \frac{\left(\frac{\pi}{3}\right)^2}{2!} = 0.45169$

% error: $\left|\frac{0.45169-0.5}{0.5}\right| \times 100 = 9.7\%$

(b) $1 - \frac{\left(\frac{\pi}{6}\right)^2}{2!} + \frac{\left(\frac{\pi}{6}\right)^4}{4!} = 0.86605$

% error: $\left|\frac{0.86605-0.86603}{0.86603}\right| \times 100 = 0.0023\%$

$1 - \frac{\left(\frac{\pi}{3}\right)^2}{2!} + \frac{\left(\frac{\pi}{3}\right)^4}{4!} = 0.50180$

% error: $\left|\frac{0.50180-0.5}{0.5}\right| \times 100 = 0.36\%$ (sic)

(ii)(a) $e^1 = 2.71828$; $1 + 1 + \frac{1^2}{2!} = 2.5$

% error: $\left|\frac{2.5-2.71828}{2.71828}\right| \times 100 = 8.0\%$

$e^2 = 7.38906$; $1 + 2 + \frac{2^2}{2!} = 5$

% error: $\left|\frac{5-7.38906}{7.38906}\right| \times 100 = 32\%$

(b) $1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} = 2.70833$

% error: $\left|\frac{2.70833-2.71828}{2.71828}\right| \times 100 = 0.37\%$

$1+2+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}=7$

% error: $\left|\frac{7-7.38906}{7.38906}\right| \times 100 = 5.3\%$

(iii)(a) $\ln(1+0.1) = 0.095310;\ 0.1-\frac{1}{2}\ (0.1)^2 = 0.095$

% error: $\left|\frac{0.095-0.095310}{0.095310}\right| \times 100 = 0.33\%$

$\ln(1+0.5) = 0.40547;\ 0.5-\frac{1}{2}\ (0.5)^2 = 0.375$

% error: $\left|\frac{0.375-0.40547}{0.40547}\right| \times 100 = 7.5\%$

$\ln(1+1) = 0.69315;\ 1-\frac{1}{2}\ (1)^2 = 0.5$

% error: $\left|\frac{0.5-0.69315}{0.69315}\right| \times 100 = 28\%$

(b) $0.1-\frac{1}{2}\ (0.1)^2+\frac{1}{3}(0.1)^3-\frac{1}{4}(0.1)^4 = 0.095308$

% error: $\left|\frac{0.095308-0.095310}{0.095310}\right| \times 100 = 0.0021\%$

$0.5-\frac{1}{2}\ (0.5)^2+\frac{1}{3}(0.5)^3-\frac{1}{4}(0.5)^4 = 0.40104$

% error: $\left|\frac{0.40104-0.40547}{0.40547}\right| \times 100 = 1.1\%$

$1-\frac{1}{2}\ (1)^2+\frac{1}{3}(1)^3-\frac{1}{4}(1)^4 = 0.58333$

% error: $\left|\frac{0.58333-0.69315}{0.69315}\right| \times 100 = 16\%$

[Note that we are at the extreme end of the range of validity.]

(5***) Use the 1st 5 terms of a Maclaurin expansion to find an approximate value for $P(Z < 1)$, where $Z \sim N(0,1)$ and Z has pdf

$f(z) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}z^2)$

Solution

$$P(Z < 1) = 0.5 + \int_0^1 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) dz$$

$$\approx 0.5 + \frac{1}{\sqrt{2\pi}} \int_0^1 1 + \left(-\frac{1}{2}z^2\right) + \frac{\left(-\frac{1}{2}z^2\right)^2}{2!} + \frac{\left(-\frac{1}{2}z^2\right)^3}{3!} + \frac{\left(-\frac{1}{2}z^2\right)^4}{4!} \; dz$$

$$= 0.5 + \frac{1}{\sqrt{2\pi}} \left[z - \frac{z^3}{6} + \frac{z^5}{40} - \frac{z^7}{336} + \frac{z^9}{3456}\right]_0^1$$

$$= 0.5 + \frac{1}{\sqrt{2\pi}} (1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456})$$

$$= 0.5 + \frac{1}{\sqrt{2\pi}} (0.8556465) = 0.84135$$

[Normal tables give 0.8413]

(6***) Use 3 terms of a Maclaurin expansion of $ln\left(\frac{1+x}{1-x}\right)$ to find an approximate value for $\ln(\frac{2}{3})$

Solution

$$ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$= \left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right\}$$

$$- \left\{[-x] - \frac{[-x]^2}{2} + \frac{[-x]^3}{3} - \frac{[-x]^4}{4} + \frac{[-x]^5}{5} - \cdots\right\}$$

$$= 2\{x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\}$$

(valid, provided that $-1 < x \leq 1$ and $-1 < -x \leq 1$;

ie $-1 < x \leq 1$ and $1 > x \geq -1$

ie $-1 < x < 1$)

Suppose that $\frac{1+x}{1-x} = \frac{2}{3}$

Then $3 + 3x = 2 - 2x$, so that $5x = -1$ and $x = -\frac{1}{5}$

(and this is within the limits of validity).

So $ln\left(\frac{2}{3}\right) \approx 2\left\{\left[-\frac{1}{5}\right] + \frac{\left[-\frac{1}{5}\right]^3}{3} + \frac{\left[-\frac{1}{5}\right]^5}{5}\right\} = -0.40546 = -0.405$ (3sf)

[The true value of $ln\left(\frac{2}{3}\right)$ is -0.40547 (5sf). Note that $x = -\frac{1}{5}$ is closer to the value of 0 (about which the Maclaurin expansion is centred) than $x = \frac{1}{3}$ [giving $\ln(1-\frac{1}{3})$], so that greater accuracy is to be expected.]

(7***) Find the first 3 non-zero terms, as well as the general term in the Maclaurin expansion of $cosh^3x$

Solution

Let $f(x) = cosh^3x$ $\quad [f(0) = 1]$

Then $f'(x) = 3cosh^2xsinhx$ $\quad [f'(0) = 0]$

and $f''(x) = 6coshxsinh^2x + 3cosh^3x$ $\quad [f''(0) = 3]$

$= 6coshx(cosh^2x - 1) + 3cosh^3x = 9cosh^3x - 6coshx$

$= 9f(x) - 6coshx$

Then $f'''(x) = 9f'(x) - 6sinhx$ $\quad [f'''(0) = 0]$

and $f^{(4)}(x) = 9f''(x) - 6coshx$

$= 81f(x) - 6coshx(1 + 9)$ $\quad [f^{(4)}(0) = 81 - 60 = 21]$

Clearly $f^{(2r+1)}(0) = 0$ and hence $a_{2r+1} = 0$

$f^{(6)}(x) = 81f''(x) - 6coshx(1 + 9)$

$= 81(9)f(x) - 6coshx(1 + 9 + 81)$

Thus $f^{(2r)}(x) = 3^{2r}f(x) - 6coshx(1 + 9 + \cdots + 9^{r-1})$

and so $a_{2r} = \frac{f^{(2r)}(0)}{(2r)!} = \frac{1}{(2r)!}\{3^{2r} - 6\frac{(9^r-1)}{(9-1)}\}$

$= \frac{1}{(2r)!}\left\{3^{2r} - 3\frac{(3^{2r}-1)}{4}\right\}$

$= \frac{3^{2r}+3}{4(2r)!}$

So $cosh^3x = 1 + \frac{3x^2}{2!} + \frac{21x^4}{4!} + \cdots + \frac{(3^{2r}+3)x^{2r}}{4(2r)!} + \cdots$

$= 1 + \frac{3x^2}{2} + \frac{7x^4}{8} + \cdots + \frac{(3^{2r}+3)x^{2r}}{4(2r)!} + \cdots$

Note: In other cases, such as $cosh^2x$, if only required to find the first few terms of the expansion, it may well be easier to expand

$\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!}\right)\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!}\right)$