Notes & Sol'ns for Q1-5 of the Specimen 2 (Mar. 2009) Paper (7 pages; 6/11/16)

(to be read in conjunction with the official solutions)

Q1/D

[Typo in question: "(d) .9" should read "(d) 9" (presumably)]

We can in fact 'cheat' by just plugging in the possible answers, to show that 7 works, but 8 and 9 don't.

Q1/G (Sol'n)

The fact that the length of N is a large number (ie 100) simplifies the problem, as there can be no question of having to list the possibilities. The solution must depend on some pattern existing.

The two digit multiples of 13 are 13, 26, 39, 52, 65, 78 & 91

(which doesn't eliminate any of the suggested answers).

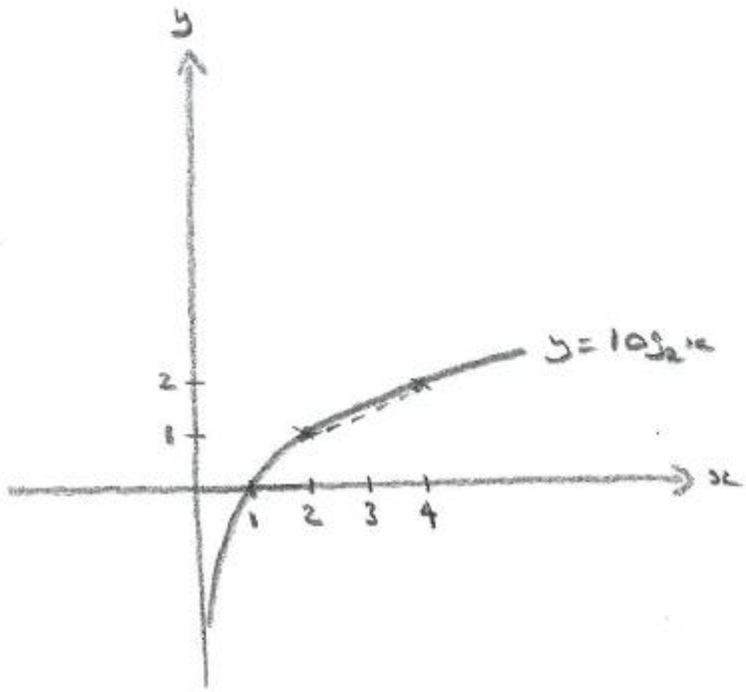
Given that the 1st digit is 9, the 2nd digit must be 1; the 3rd digit 3, and the 4th digit 9, so that we have the cycle 913. This accounts for the first 99 digits, so that the last digit must be 9.

Q1/I

To make this less of a mind-reading exercise, we can of course use any method we please, as $log_2 3$ is in only one of the given intervals, (and it's a multiple choice question); ie we're not forced to justify a particular interval using only the given information (and nothing else).

A lower bound of 1.5 can be established by linear interpolation:

As $log_2 2 = 1$ and $log_2 4 = 2$, $log_2 3 > \frac{1}{2}(1+2) = 1.5$



For the upper bound, though, it isn't obvious what the examiners had in mind.

It turns out that the idea (more generally) is to find integers a, b and c such that

$3^a < 2^b$, so that $a log_2 3 < b$,

where we want 3^a and 2^b to be as close as possible.

So, as (almost) prompted in the question, we have $3^3 < 2^5$.

Then $3 log_2 3 < 5$, and so $log_2 3 < 1\frac{2}{3}$, making the answer (b).

The method also provides a lower bound (which in the question is just the 1.5 that can be found by linear interpolation).

So we have $2^3 < 3^2$, giving $3 < 2 log_2 3$ and $log_2 3 > 1.5$

Q2 (Sol'n)

(i) Equating coefficients:

$x^3: 0 = -a + a$

$x^2: A = b - a^2 + b$

$x: 0 = ab - ba$

$x^0: B = b^2$

So $B = b^2$ & $A = 2b - a^2$

(ii) $B = 16 = b^2$ (1)

& $A = -20 = 2b - a^2$ (2)

(2) $\Rightarrow 4b^2 = (a^2 - 20)^2$

(though there may be a spurious solution)

and then (1) $\Rightarrow 64 = a^4 - 40a^2 + 400$

$\Rightarrow a^4 - 40a^2 + 336 = 0$

$\Rightarrow a^2 = \frac{40 \pm \sqrt{1600 - 4(336)}}{2} = 20 \pm \sqrt{400 - 336} = 20 \pm 8$

So $a = \pm\sqrt{28}$ & $b = \frac{1}{2}(28 - 20) = 4$, from (2) (3)

or $a = \pm\sqrt{12}$ & $b = \frac{1}{2}(12 - 20) = -4$ (4)

These solutions all agree with (1)& (2); ie there are no spurious solutions.

So the required product is either

$(x^2 + \sqrt{28}x + 4)(x^2 - \sqrt{28}x + 4)$

or $(x^2 + \sqrt{12}x - 4)(x^2 - \sqrt{12}x - 4)$

[The official sol'ns omit the 2nd possibility, but it can easily be expanded to obtain $x^4 - 20x^2 + 16$]

(iii) **Method 1a**

From the factorisation $(x^2 + \sqrt{28}x + 4)(x^2 - \sqrt{28}x + 4)$,

the 1st factor gives

$x = \frac{-\sqrt{28} \pm \sqrt{28-16}}{2} = -\sqrt{7} \pm \sqrt{3}$,

and the 2nd factor gives $x = \frac{\sqrt{28} \pm \sqrt{28-16}}{2} = \sqrt{7} \pm \sqrt{3}$,

as required

Method 1b

From the factorisation $(x^2 + \sqrt{12}x - 4)(x^2 - \sqrt{12}x - 4)$,

the 1st factor gives

$x = \frac{-\sqrt{12} \pm \sqrt{12+16}}{2} = -\sqrt{3} \pm \sqrt{7}$,

and the 2nd factor gives $x = \frac{\sqrt{12} \pm \sqrt{12+16}}{2} = \sqrt{3} \pm \sqrt{7}$

as required

Method 2a

$(x + \sqrt{7} + \sqrt{3})(x - \sqrt{7} - \sqrt{3})(x + \sqrt{7} - \sqrt{3})(x - \sqrt{7} + \sqrt{3})$

$= (x^2 - (\sqrt{7} + \sqrt{3})^2)(x^2 - (\sqrt{7} - \sqrt{3})^2)$

$= x^4 + Cx^2 + D$,

where $C=-\left(\sqrt{7}-\sqrt{3}\right)^2-\left(\sqrt{7}+\sqrt{3}\right)^2$

and $D=\left(\sqrt{7}+\sqrt{3}\right)^2\left(\sqrt{7}-\sqrt{3}\right)^2$

Then $C=-\left(7+3+2\sqrt{21}+7+3-2\sqrt{21}\right)=-20$

and $D=(7-3)^2=16$, as required

Method 2b

$\left(x+\sqrt{7}+\sqrt{3}\right)\left(x+\sqrt{7}-\sqrt{3}\right)$

$=x^2+x\left(\sqrt{7}-\sqrt{3}+\sqrt{7}+\sqrt{3}\right)+(7-3)$

$=x^2+\sqrt{28}x+4$

and $\left(x-\sqrt{7}+\sqrt{3}\right)\left(x-\sqrt{7}-\sqrt{3}\right)$

$=x^2+x\left(-\sqrt{7}-\sqrt{3}-\sqrt{7}+\sqrt{3}\right)+(7-3)$

$=x^2-\sqrt{28}x+4$

and we have already established in (ii) that

$\left(x^2+\sqrt{28}x+4\right)\left(x^2-\sqrt{28}x+4\right)=x^4-20x^2+16$

Method 2c

$\left(x+\sqrt{7}+\sqrt{3}\right)\left(x-\sqrt{7}+\sqrt{3}\right)$

$=x^2+x\left(-\sqrt{7}+\sqrt{3}+\sqrt{7}+\sqrt{3}\right)-(7-3)$

$=x^2+\sqrt{12}x-4$

and $\left(x+\sqrt{7}-\sqrt{3}\right)\left(x-\sqrt{7}-\sqrt{3}\right)$

$=x^2+x\left(-\sqrt{7}-\sqrt{3}+\sqrt{7}-\sqrt{3}\right)-(7-3)$

$= x^2 - \sqrt{12}x - 4$

and we know that

$(x^2 + \sqrt{12}x - 4)(x^2 - \sqrt{12}x - 4) = x^4 - 20x^2 + 16$

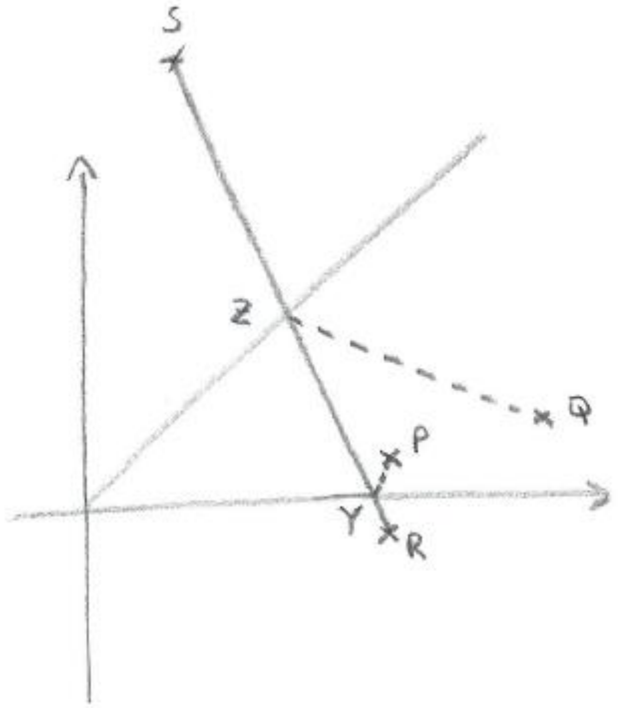
Q4

For (iv), it's a reasonable assumption that S will feature in the solution - noting that, for any point Z on the line l, $SZ = ZQ$.

It isn't always clear with this sort of question whether we are expected to employ the same idea as in (ii), but extended in some way; or whether the actual result from (ii) is to be used. Both approaches may need to be considered. Often the 2nd approach can be easier to apply, whilst the 1st requires some inspiration (assuming, of course, that there is anything to be discovered).

As an example of this (in a different context), the formula for the distance between two points in 3 dimensions is an extension of Pythagoras, but also (as it happens) Pythagoras' theorem in 2 dimensions can be used to derive it.

We want to find the shortest path PYZQ. This will have the same length as PYZS. Z can of course be anywhere on the line l, which means that we just need to find the shortest path PYS. This is the same problem as in (ii), but with S instead of Q. So we will be using R again, and the required straight line, having the same length as PYS, is RS.



The question doesn't make it clear what exactly is required: it says "Find the shortest such path", so perhaps we need to establish the coordinates of Z. The official solution just gives the length of RS - which it would probably be wise to determine anyway.