Notes & Sol'ns for Q1-5 of the Specimen 1 (created in Mar. 2009) Paper (8 pages; 26/10/16)

(to be read in conjunction with the official solutions)

Q1/A -

Q1/B

To get a feel for the problem, we can establish that $f(1) = 8$ and $f(2) = 7$.

Another thing that can be done quickly is to find $f'(x)$, which should enable us to do a rough sketch of the function in the range $0 \leq x \leq 2$.

Thus $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2)$,

which conveniently factorises to $6(x-1)(x-2)$.

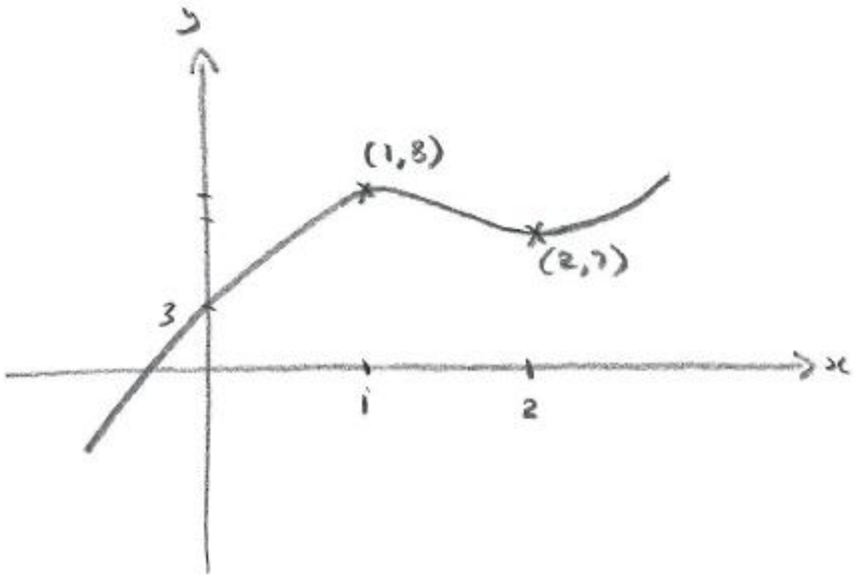
[This is a typical feature of MAT problems: only when a fairly obvious line of investigation is pursued do we discover a simplifying feature.]

From the graph of $y = 6(x-1)(x-2)$,

[or just $y = (x-1)(x-2)$] we see that $f'(x) > 0$ for $0 \leq x < 1$;

$f'(1) = 0$; $f'(x) < 0$ for $1 < x < 2$, and $f'(2) = 0$

So a rough sketch of $f(x)$ is as follows:



and the answer is seen to be 3; ie (b).

Q1/C

The potentially time-consuming solution is made simpler by the fact that the normal line through (3,4) is found to be a line through the Origin (ie simplifying the solution of the simultaneous equations).

Q1/D -

Q1/E -

Q1/F -

Q1/G

Until we attempt to factorise the associated quadratic expressions, it isn't clear how best to proceed. But having established that we require both $(x+2)(x+1)>0$ and $(x+2)(x-1)<0$, we can separately consider the 4 cases $x<-2$, $-2<x<-1$, $-1<x<1$ & $x>1$ (since

$x = -2, -1$ or 1 will not give a strict inequality).

Q1/H

Note first of all that (a)-(d) are mutually exclusive, so it isn't a matter of finding a result that is deduced from the given information only; ie we could use any method if we wanted (as it's multiple choice).

The natural thing to do with a logarithmic equation such as $log_{10}2 = 0.3010$ is to convert it into an exponential equation, even if we can't be sure that it will lead anywhere.

Thus $2 = 10^{0.3010}$, and then $2^{100} = 10^{30.10} = 10^{30} \times 10^{0.10}$

Then $10^{0.10} > 10^0 = 1$

and $10^{0.10} < 10^{0.2} < 2$

(in both cases, even allowing for the original rounding to 4dp).

Noting that $100 < 10^2 \times 10^{0.10} < 200$,

we can then say that $10^{30.10}$ (and hence 2^{100}) must have 31 digits and begin with a 1, so that the answer is (c).

Q1/I

By starting to write out the coefficients of x^0, x^1 ...

(call these c_0, c_1 ...) we see that

$\frac{c_1}{c_0} = 10\left(\frac{1}{2}\right)$, $\frac{c_2}{c_1} = \frac{9}{2}\left(\frac{1}{2}\right)$, $\frac{c_3}{c_2} = \frac{8}{3}\left(\frac{1}{2}\right)$

Thus $\frac{c_3}{c_2} > 1$, as 8 exceeds twice 3, and it is going to be worth our while to continue this process 'manually':

$\frac{c_4}{c_3} = \frac{7}{4}\left(\frac{1}{2}\right) < 1$, and so c_3 is the greatest coefficient.

(Had it taken longer for the ratio to fall below 1, then the formula approach in the official solution would be necessary.)

Q1/J

Having eliminated (a) and (b), as in the official solution, we could eliminate (d) by noting that, if $x < 0$ & $y < 0$; say $x = -a^2$

& $y = -b^2$, then $-a^4b^4(a^2 + b^2) = 1$, which has no solutions.

Alternatively, as $x \to 0$, $y^2(x + y) = \frac{1}{x^2} \to \infty$, requiring $y \to \infty$ (this being the case for positive or negative x), which is consistent with (c), but not (d).

Q3

(i) $f'(x) = 0 \Rightarrow 2x - 2p = 0 \Rightarrow x = p$

So there will be a stationary value in the range $0 < x < 1$ if and only if $0 < p < 1$.

(ii) [It may be worth doing some rough sketches of possible configurations of $y = f(x)$ at this point.]

As we have a u-shaped curve, with a minimum at $x = p \geq 1$,

$m = f(1) = 1 - 2p + 3 = 4 - 2p$

(iii) As the minimum is at $x = p \leq 0, m = f(0) = 3$

(iv) As the minimum is at $x = p,\ m = f(p) = p^2 - 2p^2 + 3$

$= 3 - p^2$

(v) For $-2 \leq p \leq 0, m = 3$

For $0 < p < 1, m = 3 - p^2$ (an n-shaped quadratic)

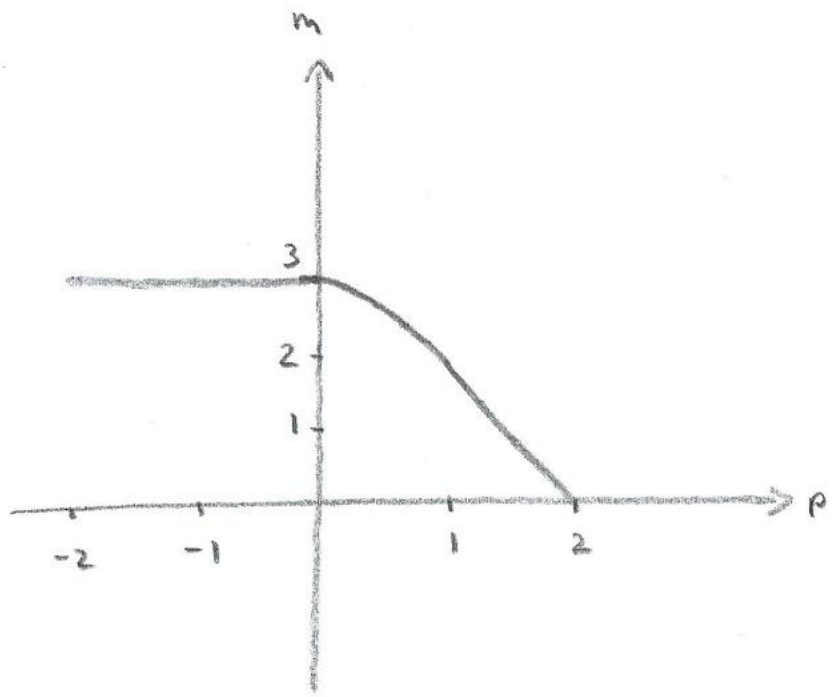
For $p \geq 1, m = 4 - 2p$

[We can see that these functions agree at the points where they join. We might reasonably expect the gradients to agree as well, so to check this:

$\frac{d}{dp}(3 - p^2) = -2p = 0$, when $p = 0$

And $-2p = -2$, when $p = 1$, which is the gradient of $m = 4 - 2p$.

Note though that the 2nd derivatives don't agree.]



Q4

(i) Alternatively, to show that the area is $\frac{1}{2}bcsin\alpha$:

$sin\alpha = \frac{QB}{c}$ and area $= \frac{1}{2}QB.b = \frac{1}{2}b(csin\alpha)$

The proof in the official solution arguably doesn't quite represent the thought processes involved for someone who hasn't tackled the problem before: although AB could be chosen as the base as part of an experimental approach, the real reason for chosing it is that the height to be used is then CR, and this can be expressed in terms of $sin\alpha$.

(ii) The form of the answer suggests subtracting Area(AQR)+Area(BPR)+Area(CQP) from Area(ABC), and symmetry suggests that we should try to show that

Area(AQR) $= cos^2\alpha \times Area(ABC)$

(iii) In order that Area(PQR) $= 0$, P, Q & R must lie on a straight line. A bit of experimenting shows that ABC has to be right-angled for this to happen (assuming that Area(ABC) > 0).

Q5

(i) [The wording seems to be slightly ambiguous: presumably it means that a y is added after the two repetitions (rather than requiring that the last note of the (2nd) repetition is a y).]

From I & II, xy is a song, and then from III, yx is also a song.

Applying II to these two songs gives: $xxyxyy$ & $xyxyxy$

And, from III we also get $yyxyxx$ & $yxyxyx$

(ii) The 1st part follows from II & III (noting that there are no other ways of creating songs of length $2m + 2$). Strictly speaking, we should also check that none of these songs are the same: there will be k different songs resulting from applying II, and then the further k songs resulting from applying III will be new ones, because they all end in an x (whereas the 1st batch of k songs all end in a y).

For the 2nd part:

To clarify: $n \rightarrow 2^{n+1} - 2$ (length) $\rightarrow 2^n$ (number of songs of that length); ie not all lengths will be possible.

This can be tackled by induction:

First of all, we show that the result is true for $n = 1$:

There are $2^1 = 2$ songs of length $2^2 - 2 = 2$ (namely xy & yx).

[Note that the natural numbers start at 1 (this is the usual definition; in some countries, they include 0); the question presumably meant to say "... for each natural number n"]

Then assume that there are 2^k songs of length $2^{k+1} - 2$ (*)

rtp [result to prove]: there are 2^{k+1} songs of length $2^{k+2} - 2$ (**)

1st part of (ii) & (*) ⇒ there are $2(2^k)$ songs of length

$2(2^{k+1} - 2) + 2 = 2^{k+2} - 2$, as required

If the result is true for $n = 1$, then from (**) it will be true for

$n = 2, 3, \ldots$ and hence all n, by the principle of induction.

Alternative method:

As n is increased by 1, the new length L_n is related to the previous one by $L_n = 2L_{n-1} + 2$ and the number of songs is multiplied by 2 (from the 1st part of (ii)).

So we see that

$L_1 = 2,\ L_2 = 2^2 + 2,\ L_3 = 2(2^2 + 2) + 2 = 2 + 2^2 + 2^3,$

and so on, giving

$L_n = 2 + 2^2 + 2^3 + \cdots + 2^n = \frac{2(2^n - 1)}{2-1} = 2^{n+1} - 2$, as required.

(iii) [Remember that MAT questions don't usually involve anything too obscure, so it's worth considering simple outcomes.]

For the songs from the "Classical period", the possible lengths were governed entirely by the $2m + 2$ rule, and each possible length only gave rise to another length of higher value. With the "later period" songs, we can now go backwards as well - potentially making the situation very complicated.

However, we note that:

(a) For any given length, we can always produce a length of greater value.

(b) To cover any missed values, we can always subtract one: since this is allowed by IV if the last note is a y; and if it is an x, then we can obtain a song of the same length, ending in a y, by III - and then subtract one from that.

ie we can make a song of any length.