MAT: Instructive Questions & Exercises

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19. The positive real numbers a, b, and c are such that the equation

$$x^3 + ax^2 = bx + c$$

has three real roots, one positive and two negative.

Which one of the following correctly describes the real roots of the equation

$$x^3 + c = ax^2 + bx\ ?$$

A It has three real roots, one positive and two negative.

B It has three real roots, two positive and one negative.

C It has three real roots, but their signs differ depending on a, b, and c.

D It has exactly one real root, which is positive.

E It has exactly one real root, which is negative.

F It has exactly one real root, whose sign differs depending on a, b, and c.

G The number of real roots can be one or three, but the number of roots differs depending on a, b, and c.

Solution

[A graphical approach looks promising here, but unfortunately turns out to be too complicated.]

The two equations can be rewritten as:

$x^3 + ax^2 - bx - c = 0$ (1)

and $x^3 - ax^2 - bx + c = 0$ (2)

[Given that only the signs of even powers of x differ]

Let $y = -x$

Then (2) becomes $-y^3 - ay^2 + by + c = 0$

or $y^3 + ay^2 - by - c = 0$, which has the same roots as (1).

So, if (1) has one positive and two negative roots, (2) will have one negative and two positive roots.

Answer: B

Comments

Example of rearrangement (substitution)[after observing that only the signs of even powers of x differ]

(2) MAT 2007, Q1(D) [Geometry]

D. The point on the circle

$$(x-5)^2+(y-4)^2=4$$

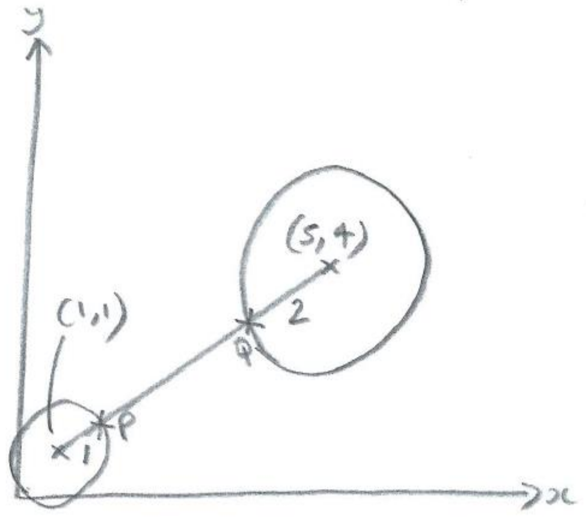
which is closest to the circle

$$(x-1)^2+(y-1)^2=1$$

is

(a) $(3.4, 2.8)$, (b) $(3, 4)$, (c) $(5, 2)$, (d) $(3.8, 2.4)$.

Solution



The distance between the two centres is 5 (by Pythagoras), and the required point is $\frac{2}{5}$ of the way along the line joining the centres, from the point (5,4).

Taking a weighted average of the two centres [' linear interpolation']:

$\frac{2}{5}(1,1)+\frac{3}{5}(5,4)=(\frac{17}{5},\frac{14}{5})$ or (3.4, 2.8)

So the answer is (a).

Comments

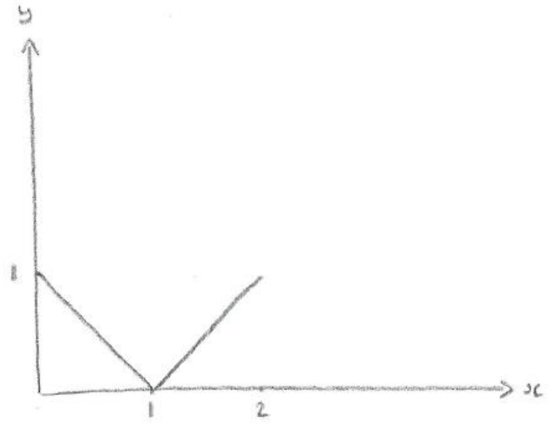
Simplifying features (3,4,5 triangle) emerges only after diagram has been drawn.

Example of use of linear interpolation.

(3) Sketch the graph of $\sqrt{x^2 - 2x + 1}$ for $0 \leq x \leq 2$

[Curve sketching]

Solution



For $0 \leq x \leq 1$, $\sqrt{x^2 - 2x + 1} = \sqrt{(x-1)^2} = \sqrt{(1-x)^2} = 1 - x$

For $1 \leq x \leq 2$, $\sqrt{x^2 - 2x + 1} = \sqrt{(x-1)^2} = x - 1$

Comments

Example of Case by Case approach.

(4) How many solutions are there to $x^3 - 6x^2 + 9x + 2 = 0$?

[Polynomials]

Solution

$\Leftrightarrow x(x^2 - 6x + 9) = -2$

$\Leftrightarrow x(x-3)^2 = -2$

So one solution, from graph of $y = x(x-3)^2$

Comments

Example of rearrangement, and reformulation of problem (as graphical rather than algebraic problem).

(5) MAT, Specimen Paper B, Q1/G [Digits]

G. The four digit number 2652 is such that any two consecutive digits from it make a multiple of 13. Another number N has this same property, is 100 digits long, and begins in a 9. What is the last digit of N?

(a) 2 (b) 3 (c) 6 (d) 9

Solution

The two digit multiples of 13 are 13, 26, 39, 52, 65, 78 & 91

(which doesn't eliminate any of the suggested answers).

Given that the 1st digit is 9, the 2nd digit must be 1; the 3rd digit 3, and the 4th digit 9, so that we have the cycle 913. This accounts for the first 99 digits, so that the last digit must be 9.

ie the answer is (d)

Comments

Look for something that is quick to do (and likely to be worthwhile); ie consider multiples of 13.

“100 digits long”: suggests looking for a pattern

(6) MAT 2009, Q1/D [Integers / Series]

D. The smallest positive integer n such that

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1}\, n \geqslant 100,$$

is

(a) 99, (b) 101, (c) 199, (d) 300.

Solution

[Because of the presence of $(-1)^{n+1}$, it is worth considering separately even and odd n.]

With even n, the LHS becomes $1-2+3-4+\cdots-2m$, writing $n=2m$.

By grouping the terms as $(1-2)+(3-4)+\cdots-2m$, we see that this has a negative value.

So n must be odd, and the LHS becomes

$1-2+3-4+\cdots+(2m+1)$, writing $n=2m+1$

And the terms can be grouped to give

$(1-2)+(3-4)+\cdots([2m-1]-2m)+(2m+1)$

$=m(-1)+(2m+1)=m+1$

So we want $m+1\geq 100$, and hence

$n=2m+1\geq 2(99)+1=199$

So the answer is (c).

Comments

Example of Case by Case.

Experimenting: eliminates even n

(7) Find all positive integer solutions of the equation

$xy - 8x + 6y = 90$ [Integers]

Solution

$xy - 8x + 6y = (x + 6)(y - 8) + 48,$

so that the original equation is equivalent to

$(x + 6)(y - 8) = 42$

The positive integer solutions are given by:

$x + 6 = 7, y - 8 = 6$

$x + 6 = 14, y - 8 = 3$

$x + 6 = 21\ y - 8 = 2$

$x + 6 = 42, y - 8 = 1,$

so that the solutions are:

$x = 1, y = 14$

$x = 8, y = 11$

$x = 15, y = 10$

$x = 36, y = 9$

Comments

- Consider a simpler problem; eg $xy = 90$
- Standard idea for integer-related questions is factorisation.
- Re-read question: “Find all **positive** integer solutions”.
- Consider ALL cases (eg $x + 6 = -7, y - 8 = -6$)

(8) Can n^3 equal $n + 12345670$ (where n is a positive integer)?

[Integers]

Solution

Rearrange to $n^3 - n = 12345670$

$n^3 - n = n(n^2 - 1) = n(n - 1)(n + 1)$

One of these factors must be a multiple of 3; whereas 12345670 is not a multiple of 3 (since $1 + 2 + 3 + 4 + 5 + 6 + 7 + 0$ isn't a multiple of 3); so answer is No.

Comments

- Large number means that trial and error can probably be ruled out.

- Example of rearrangement.

- Standard idea for integer-related questions is factorisation.

- Useful idea (division by 3) only emerges once you have experimented.

(9) MAT 2007, Q1/J [Inequality / Series]

J. The inequality

$$(n+1)+(n^4+2)+(n^9+3)+(n^{16}+4)+\cdots+(n^{10000}+100)>k$$

is true for all $n \geqslant 1$. It follows that

(a) $k<1300$,
(b) $k^2<101$,
(c) $k \geqslant 101^{10000}$,
(d) $k<5150$.

Solution

Consider $n = 1$ [as this is fairly quick to do]

The LHS is $100 + \frac{1}{2}(100)(101) = 5150$

As the LHS increases with n, the inequality will hold provided that $k < 5150$

So the answer is (d).

Comments

- Experiment with a particular value.

(10) MAT 2008, Q1/B [Logarithms]

B. Which is the smallest of these values?

(a) $\log_{10}\pi$, (b) $\sqrt{\log_{10}(\pi^2)}$, (c) $\left(\dfrac{1}{\log_{10}\pi}\right)^3$, (d) $\dfrac{1}{\log_{10}\sqrt{\pi}}$.

Solution

Write $L = log_{10}\pi$

[It often helps to have a rough idea of the sizes of the multiple choice options.]

$L \approx \frac{1}{2}$, so that $(b) = \sqrt{2L} \approx 1, (c) = \left(\frac{1}{L}\right)^3 = 8, (d) = \frac{1}{\frac{1}{2}L} \approx 4$

so that we can be fairly sure that the answer is (a), and might just like to check that (a) < (b):

result to prove: $L < \sqrt{2L}$

As $L < 1$ (as $\pi < 10$), $\sqrt{2L} = \sqrt{2}\sqrt{L} > \sqrt{2}\,L > L$, as required.

So the answer is (a).

Comments

Use of approximate values

(11) MAT 2009, Q1/J [Integers]

J. The number of *pairs* of *positive integers* x, y which solve the equation

$$x^3 + 6x^2y + 12xy^2 + 8y^3 = 2^{30}$$

is

(a) 0, (b) 2^6, (c) $2^9 - 1$, (d) $2^{10} + 2$.

Solution

The presence of $8y^3$ suggests that $(x + 2y)^3$ might possibly expand to give the LHS - which it does.

This then gives $x + 2y = 2^{10}$, and we can simplify matters by writing $x = 2u$ (since x has to be even), to give $u + y = 2^9$.

Then y can take the values $1, 2, \dots, 2^9 - 1$ (with $x = 2^{10} - 2y$), so that there are $2^9 - 1$ such pairs.

So the answer is (c).

Comments

Anything complicated-looking is likely to have a simple interpretation.

(12) MAT 2008, Q1/J [Trigonometry]

J. In the range $0 \leqslant x < 2\pi$ the equation

$$(3+\cos x)^2 = 4 - 2\sin^8 x$$

has

(a) 0 solutions, (b) 1 solution, (c) 2 solutions, (d) 3 solutions.

Introduction

This equation can be interpreted as the intersection of the functions $y = (3 + cosx)^2$ and $y = 4 - 2sin^8x$.

In order for the functions to intersect, their ranges must overlap.

Given the relatively complicated nature of these functions, the simplest outcome for this question would be that either the ranges don't overlap at all, or they overlap at one value.

Solution

Note that $(3 + cosx)^2 \geq 4$,

whilst $4 - 2sin^8x \leq 4$

So a sol'n only exists when $cosx = -1$ and $sinx = 0$;

ie at $x = \pi$ (in the range $0 \leq x < 2\pi$)

So the answer is (b).

(13) MAT 2008, Q1/I [Series]

2008/1I

The function $S(n)$ is defined for positive integers n by

$$S(n) = \text{sum of the digits of } n.$$

For example, $S(723) = 7 + 2 + 3 = 12$.

The sum $S(1) + S(2) + S(3) + \cdots + S(99)$

equals

(a) 746 (b) 862 (c) 900 (d) 924

Solution

[The official solution is based on spotting the 'lateral thinking' idea that, by symmetry, there must be 20 of each of the 10 digits

0, 1, 2, ..., 9 amongst the numbers 00, 01, ..., 99 (there being 200 digits in total). Alternative method:]

$S(1) + \cdots + S(9) = \sum_{i=1}^{9} i = \frac{1}{2}(9)(10) = 45$

$S(10) + \cdots + S(19) = (10 \times 1) + \sum_{i=1}^{9} i = 10 + 45$

[not writing 55 at this stage, so as not to lose the 45, which is clearly going to be cropping up again]

$S(20) + \cdots + S(29) = (10 \times 2) + \sum_{i=1}^{9} i = 20 + 45$,

and so on, giving a grand total of

$(10 + 20 + \cdots + 90) + (10 \times 45) = 10(45) + 450 = 900$

So the answer is (c).

(14) TMUA, Specimen P1, Q10 [Transformations]

10. The curve $y = \cos x$ is reflected in the line $y = 1$ and the resulting curve is then translated by $\frac{\pi}{4}$ units in the positive x-direction. The equation of this new curve is

A $y = 2 + \cos\left(x + \frac{\pi}{4}\right)$

B $y = 2 + \cos\left(x - \frac{\pi}{4}\right)$

C $y = 2 - \cos\left(x + \frac{\pi}{4}\right)$

D $y = 2 - \cos\left(x - \frac{\pi}{4}\right)$

Solution

Reflecting $y = f(x)$ in the line $y = b$ can be shown to give $2b - y = f(x)$ [reflecting $y = f(x)$ in the line $x = a$ gives

$y = f(2a - x)$]

Proof: The reflection in $y = b$ is equivalent to a translation of $\begin{pmatrix} 0 \\ -b \end{pmatrix}$, followed by a reflection in the x-axis, and then a translation of $\begin{pmatrix} 0 \\ b \end{pmatrix}$: this produces $y = f(x) \rightarrow y = f(x) - b \rightarrow -(f(x) - b)$

$\rightarrow -(f(x) - b) + b = 2b - f(x)$

When $f(x) = cosx$ and $b = 1$ this gives $y = 2 - cosx$

Then a translation of $\begin{pmatrix} \frac{\pi}{4} \\ 0 \end{pmatrix}$ gives $y = 2 - \cos\,(x - \frac{\pi}{4})$,

Answer: D

(15) TMUA, Specimen P1, Q13 [Polynomials]

13. How many real roots does the equation $x^4 - 4x^3 + 4x^2 - 10 = 0$ have?

A 0

B 1

C 2

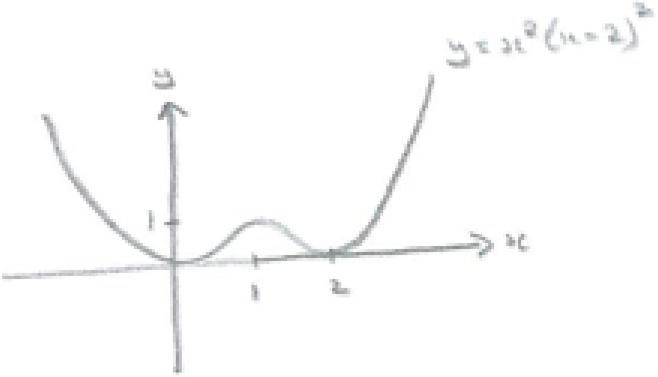
D 3

E 4

Solution

Equivalently, consider the roots of $x^2(x^2 - 4x + 4) = 10$

ie $x^2(x - 2)^2 = 10$



Referring to the graph, there are 2 roots.

[$y = f(x) = x^2(x - 2)^2$ has symmetry about $x = 1$, as the translation of $f(x)$ by 1 to the left is

$g(x) = f(x + 1) = (x + 1)^2(x - 1)^2$,

and $g(-x) = (-x + 1)^2(-x - 1)^2 = (x - 1)^2(x + 1)^2 = g(x)$,

and thus $g(x)$ is an even function (with symmetry about the y-axis)]

Answer: C

(16) TMUA 2020, P1, Q13 [Polynomials]

13 How many real roots does the equation $3x^5 - 10x^3 - 120x + 30 = 0$ have?

A 1

B 2

C 3

D 4

E 5

Solution

Writing $f(x) = 3x^5 - 10x^3 - 120x + 30$,

$f'(x) = 15x^4 - 30x^2 - 120$

Then $f'(x) = 0 \Rightarrow (x^2 - 4)(x^2 + 2) = 0 \Rightarrow x = \pm 2$

$f''(x) = 60x^3 - 60x$

$f''(-2) < 0 \Rightarrow$ maximum

$f''(2) > 0 \Rightarrow$ minimum

(and these are the only two turning points).

[The shape of a quintic means that, if $f'(x) = 0$ and $f''(x) \neq 0$ for $x = -2$ and $x = 2$, then there would have to be a maximum at $x = -2$ and a minimum at $x = 2$.]

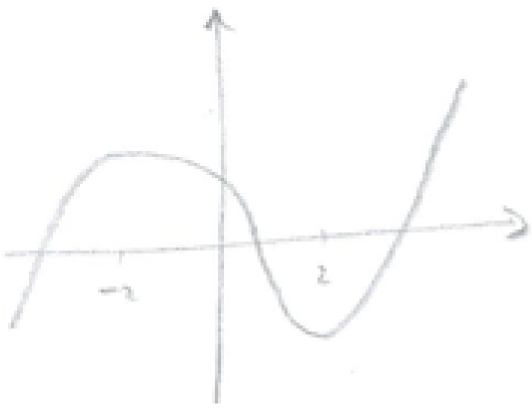
$f(-2) = 3(-32) - 10(-8) - 120(-2) + 30$

$= -96 + 80 + 240 + 30 > 0$

and $f(2) = 3(32) - 10(8) - 120(2) + 30$

$= 96 - 80 - 240 + 30 < 0$

so that the graph of $f(x)$ has the shape shown in the diagram below, and therefore there are 3 real roots of $f(x) = 0$



Answer : C

(17) TMUA 2020, P1, Q10 [Transformations]

10 The following sequence of transformations is applied to the curve $y = 4x^2$

1. Translation by $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$

2. Reflection in the x-axis

3. Stretch parallel to the x-axis with scale factor 2

What is the equation of the resulting curve?

A $y = -x^2 + 12x - 31$

B $y = -x^2 + 12x - 41$

C $y = x^2 + 12x + 31$

D $y = x^2 + 12x + 41$

E $y = -16x^2 + 48x - 31$

F $y = -16x^2 + 48x - 41$

G $y = 16x^2 - 48x + 31$

H $y = 16x^2 - 48x + 41$

Solution

Translation by $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$: $y = 4x^2 \rightarrow y = 4(x-3)^2 - 5$

Reflection in the x-axis:

$y = 4(x-3)^2 - 5 \rightarrow y = -[4(x-3)^2 - 5]$

Stretch parallel to the x-axis with scale factor 2:

$y = -[4(x-3)^2 - 5] \rightarrow y = -[4\left(\frac{x}{2} - 3\right)^2 - 5]$

$= -x^2 + 12x - 31$

Answer: A

(18) MAT 2010, Q1/E [Logarithms]

E. Which is the largest of the following four numbers?

(a) $\log_2 3$, (b) $\log_4 8$, (c) $\log_3 2$, (d) $\log_5 10$.

Solution

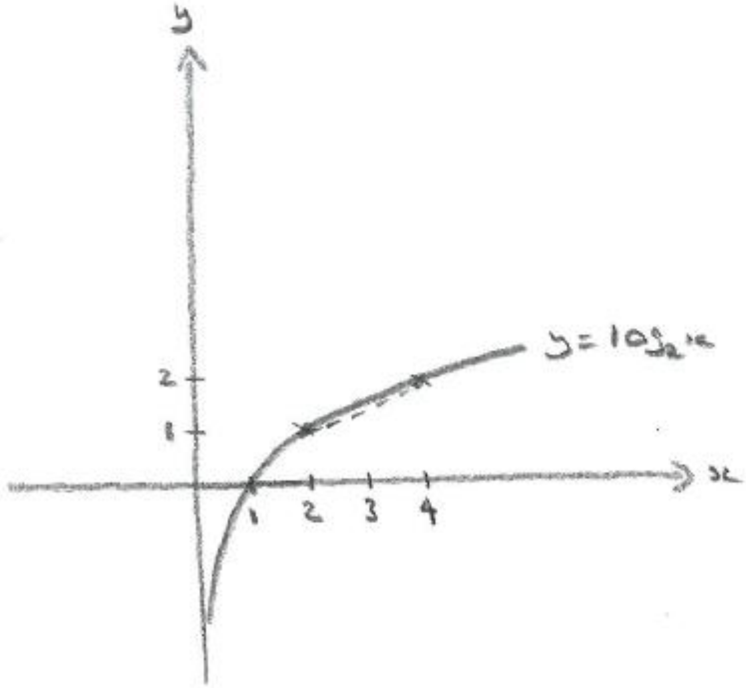
Let $log_2 3 = a$ etc

Note first of all that

$1 < a < 2, 1 < b < 2,\ 0 < c < 1$ & $1 < d < 2$, so that (c) can be eliminated.

Also $b = log_4(2^3) = 3log_4 2 = 3\left(\frac{1}{2}\right) = 1.5$

Now, from the diagram below we see that $a = log_2 3 > 1.5$



So that (b) can be eliminated.

Then $log_5 10 = log_5(5 \times 2) = log_5 5 + log_5 2 = 1 + log_5 2$

$< 1 + log_5\sqrt{5} = 1.5$, so that $d < 1.5 < a$

and **the answer is therefore (a).**

(19) MAT 2009, Q1/C [Geometry]

C. Given a real constant c, the equation

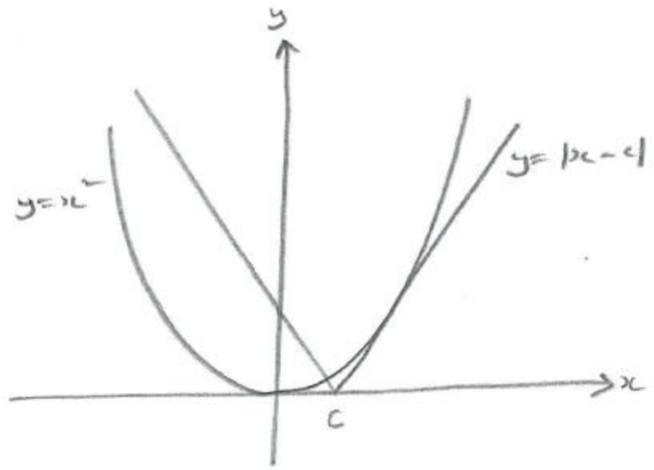
$$x^4 = (x-c)^2$$

has four real solutions (including possible repeated roots) for

(a) $c \leqslant \frac{1}{4}$, (b) $-\frac{1}{4} \leqslant c \leqslant \frac{1}{4}$, (c) $c \leqslant -\frac{1}{4}$, (d) all values of c.

Solution

$x^4 = (x-c)^2 \Rightarrow x^2 = |x-c|$



The diagram show the critical point at which the number of roots changes from 4 to 2 (for larger values of c). By symmetry, if the critical value of c is c_1, then $-c_1$ will also be a critical value (with the number of roots being 2 for $c < -c_1$. As there is only one answer of the form $-c_1 \leq c \leq c_1$,

the answer must be (b).

[As a check, the gradients of $y = x^2$ & $y = x - c$ are equal when $2x = 1$; ie $x = \frac{1}{2}$, so that the line $y = x - c$ has to pass through the point $(\frac{1}{2}, \left(\frac{1}{2}\right)^2)$, and hence $\frac{1}{4} = \frac{1}{2} - c$, giving $c = \frac{1}{4}$]

(20) TMUA 2021, P1, Q7 [Integration]

The function f is such that $f(0) = 0$, and $xf(x) > 0$ for all non-zero values of x.

It is given that

$$\int_{-2}^{2} f(x)\,dx = 4$$

and

$$\int_{-2}^{2} |f(x)|\,dx = 8$$

Evaluate

$$\int_{-2}^{0} f(|x|)\,dx$$

A -8

B -6

C -4

D -2

E 2

F 4

G 6

H 8

Solution

Let $I = \int_{-2}^{0} f(|x|)dx = \int_{-2}^{0} f(-x)dx$

Write $u = -x$, so that $I = \int_{2}^{0} f(u)(-1)du = \int_{0}^{2} f(x)dx$ (*)

As $xf(x) > 0, f(x) > 0$ when $x > 0,$ and $f(x) < 0$ when $x < 0.$

Then $\int_{-2}^{2} |f(x)|dx = 8 \Rightarrow \int_{-2}^{0} -f(x)dx + \int_{0}^{2} f(x)dx = 8$ (1)

And $\int_{-2}^{2} f(x)dx = 4$, so that $\int_{-2}^{0} f(x)dx + \int_{0}^{2} f(x)dx = 4$ (2)

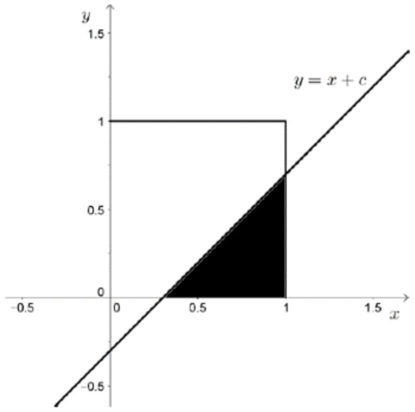
Adding (1) & (2): $2\int_{0}^{2} f(x)dx = 12,$

so that, from (*), $I = \int_{0}^{2} f(x)dx = 6$

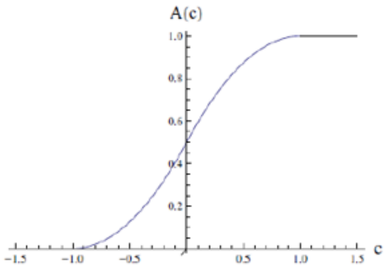
Answer: G

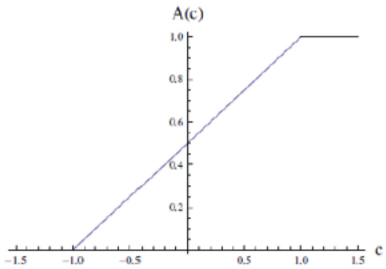
(21) MAT 2012, Q1/D [Differentiation]

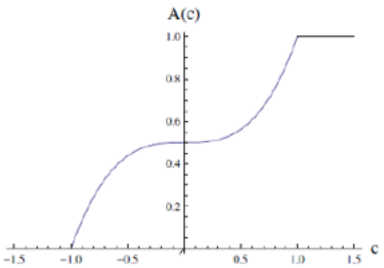
Shown below is a diagram of the square with vertices $(0,0)$, $(0,1)$, $(1,1)$, $(1,0)$ and the line $y = x + c$. The shaded region is the region of the square which lies below the line; this shaded region has area $A(c)$.

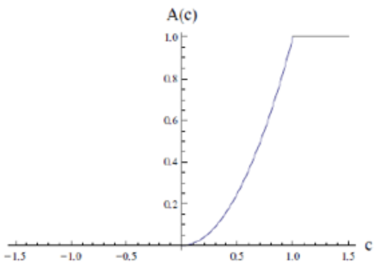


Which of the following graphs shows $A(c)$ as c varies?

(a)


(b)


(c)


(d)


Solution

A(c) increases at its greatest rate when $c = 0$, and this agrees with (a) only.

So the answer is (a).

[Alternatively: $A(0) = 0.5$, so that (d) can be eliminated.

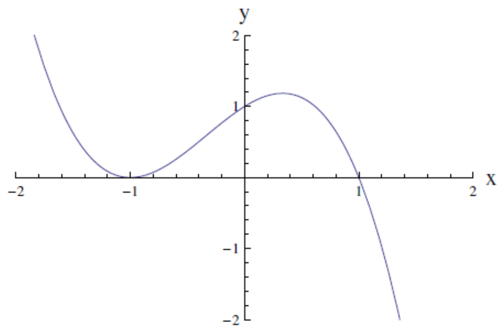
Then, for $c \leq 0$, $A(c) = \int_{-c}^{1} x + c\, dx = \left[\frac{1}{2}x^2 + cx\right]_{-c}^{1}$

$= \left(\frac{1}{2} + c\right) - \left(\frac{1}{2}c^2 - c^2\right) = \frac{1}{2}c^2 + c + \frac{1}{2}$

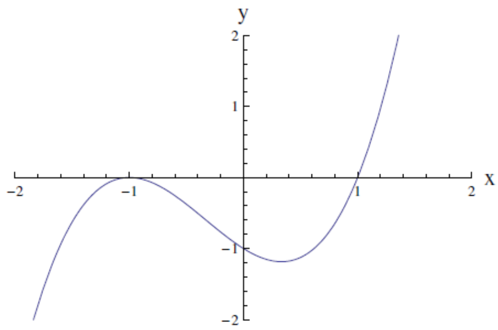
Option (b) is therefore eliminated, as it isn't a quadratic function for $c \leq 0$; whilst (c) is the wrong-shaped quadratic (being 'n-shaped', rather than 'u-shaped'). Also $A'(c) = c + 1$, so that $A'(0) = 1$, and this is inconsistent with (c), which shows a gradient of zero.]

(22) MAT 2011, Q1/A [Cubics]

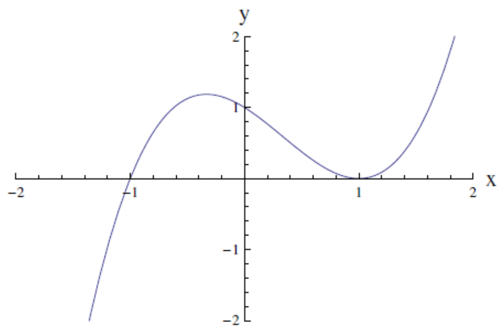
A. A sketch of the graph $y = x^3 - x^2 - x + 1$ appears on which of the following axes?



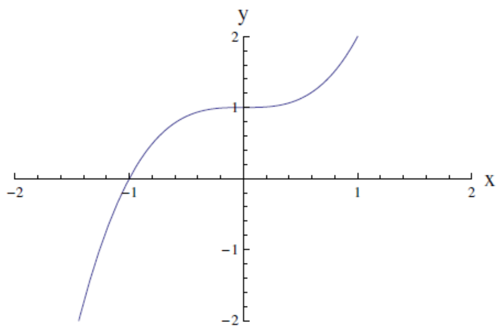
(a)



(b)



(c)



(d)

Solution

(a) starts in the wrong quadrant, and so can be eliminated.

If $f(x) = x^3 - x^2 - x + 1$,

$f'(x) = 3x^2 - 2x - 1 = (x - 1)(3x + 1)$

[Had $(x - 1)$ not been a factor, (c) could have been eliminated.]

Thus there is a stationary point at $x = 1$,

and so **the answer must be (c)**, by elimination.

[Alternative approaches:

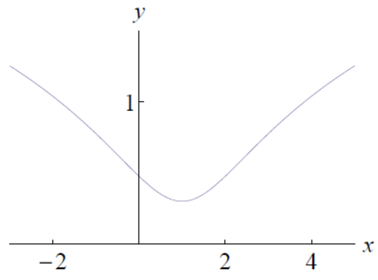
(i) Consider x-coordinate of point of inflexion $(-\frac{b}{3a} = \frac{1}{3})$

(ii) Sum of roots is expected to be $-\frac{b}{a} = 1$, which rules out (a) & (b), and is consistent with (c). However, there are two

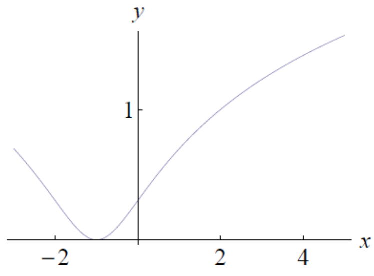
(as yet unknown) complex roots for (d).]

(23) MAT 2014, Q1/B [Curve Sketching]

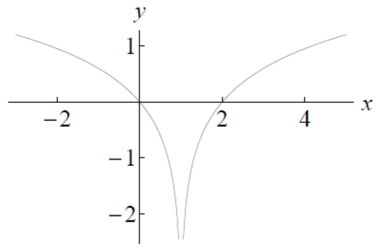
B. The graph of the function $y = \log_{10}(x^2 - 2x + 2)$ is sketched in



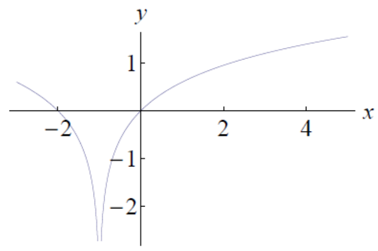
(a)



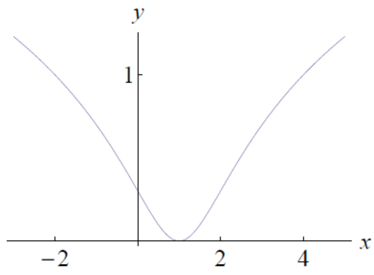
(b)



(c)



(d)



(e)

Solution

[Completing the square is something that is quick to do (and may shed some light on the problem).]

Let $f(x) = log_{10}(x^2 - 2x + 2) = log_{10}[(x-1)^2 + 1]$

Then $f(1) = 0$, so that (a), (b), (c) & (d) can be eliminated.

Answer is (e)

(24) TMUA 2021, P2, Q20 [Integration]

20 A sequence of functions f_1, f_2, f_3, … is defined by

$$f_1(x) = |x|$$

$$f_{n+1}(x) = |f_n(x) + x| \quad \text{for } n \geq 1$$

Find the value of

$$\int_{-1}^{1} f_{99}(x)\,dx$$

A 0

B 0.5

C 1

D 49.5

E 50

F 99

G 99.5

H 100

Solution

If $x \geq 0, |x| = x,$ and so $f_n(x) = nx$

If $x < 0, f_1(x) = -x, f_2(x) = |-x + x| = 0,$

$f_3(x) = |0 + x| = -x, f_4(x) = |-x + x| = 0,$ and so on

So $f_{99}(x) = -x$ (when $x < 0$).

Hence $\int_{-1}^{1} f_{99}(x)\, dx = \int_{-1}^{0} -x\, dx + \int_0^1 99x\, dx$

$= \left[-\frac{1}{2}x^2\right]_{-1}^{0} + \left[\frac{99}{2}x^2\right]_0^1$