2023 MAT – Q5 (5 pages; 27/8/25)

5. For **ALL APPLICANTS.**

Define the sequence, F_n, as follows: $F_1 = 1$, $F_2 = 1$, and for $n \geqslant 3$,

$$F_n = F_{n-1} + F_{n-2}. \qquad (*)$$

(i) [3 marks] What are the values F_3, F_4, F_5?

(ii) [1 mark] Using the equation (*) repeatedly, in terms of n, how many additions do you need to calculate F_n?

We now consider sequences of 0's and 1's of length n, that do not have two consecutive 1's. So, for $n = 5$, for example, $(0, 1, 0, 0, 1)$ and $(1, 0, 1, 0, 1)$ would be valid sequences, but $(0, 1, 1, 0, 0)$ would not. Let S_n denote the number of valid sequences of length n.

(iii) [1 mark] What are S_1 and S_2?

(iv) [3 marks] For $n \geqslant 3$, by considering the first element of the sequence of 0's and 1's, show that S_n satisfies the same equation (*). Hence conclude that $S_n = F_{n+2}$ for all n.

(v) [2 marks] For $n \geqslant 2$, by considering valid sequences of length $2n - 3$ and focusing on the element in the $(n-1)^{\text{th}}$ position, show that,

$$F_{2n-1} = F_n^2 + F_{n-1}^2. \qquad \text{(O)}$$

(vi) [3 marks] For $n \geqslant 2$, show that,

$$F_{2n} = F_n^2 + 2F_nF_{n-1}. \qquad \text{(E)}$$

(vii) [2 marks] Let $k \geqslant 3$ be an integer. By using the equations (O) and (E) repeatedly, how many arithmetic operations do you need to calculate F_{2^k}? You should only count additions and multiplications needed to calculate values using the equations (O) and (E) .

Solution

(i) $F_3 = F_2 + F_1 = 1 + 1 = 2$

$F_4 = F_3 + F_2 = 2 + 1 = 3$

$F_5 = F_4 + F_3 = 3 + 2 = 5$

(ii) To obtain F_3, the number of additions needed is 1.

To obtain F_4, the number of additions needed is 1 (to obtain F_3) $+ 1 = 2$, and so on, giving $n - 2$ additions needed to obtain F_n.

(iii) For $n = 1$ the valid sequences are (0) & (1), so $S_1 = 2$.

For $n = 2$ the valid sequences are (0,0), (0,1) & (1,0), so $S_2 = 3$.

(iv) If the 1st item of the sequence for n is 0, then the 2nd item can be 0 or 1, and the number of valid sequences will be S_{n-1}.

If the 1st item of the sequence for n is 1, then the 2nd item can only be 0, and all the valid sequences will start 10, but the 3rd item can be 0 or 1, and the number of valid sequences will then be S_{n-2}.

Thus the number of valid sequences for n can be split into 2 groups, of sizes S_{n-1} and S_{n-2} respectively; making $S_{n-1} + S_{n-2}$ in total. So S_n satisfies the equation (*) .

2nd Part

Proof by induction that $S_n = F_{n+2}$:

$S_1 = 2 = F_3$ and $S_2 = 3 = F_4$, so true for $n = 1$ & $n = 2$

Assume true for $n = k - 1$ & $n = k$, so that

$S_{k-1} = F_{k+1}$ & $S_k = F_{k+2}$

Then $S_{k+1} = S_k + S_{k-1} = F_{k+2} + F_{k+1} = F_{k+3} = F_{(k+1)+2}$,

so that, if the result is true for $n = k - 1$ & $n = k$, then it is true

for $n = k + 1$.

As it is true for $n = 1$ & $n = 2$, it is therefore true for $n = 3, 4, \dots$, and hence all positive integer n by (the principle of) Induction.

(v) [Until we use the fact that $S_n = F_{n+2}$, it isn't possible to know

the significance of the $2n - 3$ and $n - 1$ mentioned in the

instruction.]

As $S_n = F_{n+2},\ F_{2n-1} = {F_n}^2 + {F_{n-1}}^2 \Leftrightarrow S_{2n-3} = {S_{n-2}}^2 + {S_{n-3}}^2$

Noting that the $n - 1\ st$ element is the middle of the $2n - 3$

elements, which can be written as $[n - 2]X[n - 2]$,

we see that **either $\boldsymbol{X = 0}$**, in which case there are no constraints

on the adjacent elements, and the number of valid sequences is

${S_{n-2}}^2$ (for each of the S_{n-2} valid sequences for the 1st group of

$n - 2$ items, there will be S_{n-2} valid sequences for the 2nd group);

or $\boldsymbol{X = 1}$, in which case the adjacent items must both be 0, and the elements can now be written as $[n - 3]010[n - 3]$, and so the number of valid sequences is ${S_{n-3}}^2$.

Thus, the total number of valid sequences is ${S_{n-2}}^2 + {S_{n-3}}^2$, as required.

(vi) As $S_n = F_{n+2}$,

$F_{2n} = {F_n}^2 + 2F_nF_{n-1} \Leftrightarrow S_{2n-2} = {S_{n-2}}^2 + 2S_{n-2}S_{n-3}$

The elements can be written as $[n-2]XX[n-2]$.

Then, **if XX is 00**, the number of valid sequences is ${S_{n-2}}^2$.

As XX cannot be 11, there are 2 other cases to consider: 01 and 10

If XX is 01, the element following the 1 must be 0, and so we can write the elements as $[n-2]010[n-3]$, and the number of valid sequences is then $S_{n-2}S_{n-3}$.

Similarly, **if XX is 10**, the element preceding the 1 must be 0, and so we can write the elements as $[n-3]010[n-2]$, and the number of valid sequences is $S_{n-3}S_{n-2}$.

And so the total number of valid sequences is ${S_{n-2}}^2 + 2S_{n-2}S_{n-3}$, as required.

(vii) [It turns out that there is more than one way of 'using (O) and (E) repeatedly', with no way of knowing in advance which one will produce the smallest values. The method in the Official solution produces a formula of $5k-7$.) In recognition of this, the mark scheme is rather generous ("2 marks given for anything remotely sensible"! – though presumably with some reasonably sensible justification).]

$F_{2n-1} = {F_n}^2 + {F_{n-1}}^2$ (O) and $F_{2n} = {F_n}^2 + 2F_nF_{n-1}$ (E)

Let $[X]$ denote the number of arithmetic operations needed to calculate X.

Then $[F_{2^1}] = 0$ (as it is given that $F_2 = 1$)

And $[F_{2^2}] = \left[{F_2}^2 + 2F_2F_1\right]$, from (E), and this equals 4.

Then $[F_{2^3}] = \left[{F_4}^2 + 2F_4F_3\right]$, with $[F_3] = [{F_2}^2 + {F_1}^2] = 3$,

so that $[F_{2^3}] = 4(\text{for } F_4 = F_{2^2}) + 3(\text{for } \mathrm{F}_3) + 4 = 11$

Also, $[F_{2^4}] = \left[{F_8}^2 + 2F_8F_7\right]$, with $[F_7] = [{F_4}^2 + {F_3}^2] = 3$

(as F_3 & F_4 have already been calculated);

so that $[F_{2^4}] = 11 \ (\text{for } F_8) + 3(\text{for } F_7) + 4 = 18$

[which arises from $(4 + [3 + 4]) + [3 + 4]$]

And $[F_{2^5}] = \left[{F_{16}}^2 + 2F_{16}F_{15}\right]$, with $[F_{15}] = [{F_8}^2 + {F_7}^2] = 3$;

so that $[F_{2^5}] = 18 \ (\text{for } F_{16}) + 3(\text{for } F_{15}) + 4 = 25$

[which arises from $(4 + [3 + 4]) + [3 + 4] + [3 + 4]$]

So we can assert that $[F_{2^k}] = 4 + 7(k - 2) = 7k - 10$ for $k \geq 2$, and certainly for $k \geq 3$.

This could be verified by Induction.

[This question illustrates a common theme in advanced Maths. In this case, we are establishing a result concerning the Fibonacci sequence (F_n) by first of all reformulating the problem in terms of a different sequence (S_n), which is more easily used to establish the result in question.]