2023 MAT – Q3 (2 pages; 18/8/25)

3. For **ALL APPLICANTS.**

Note that the arguments of all trigonometric functions in this question are given in terms of degrees. You are not expected to differentiate such a function. The notation $\cos^n x$ means $(\cos x)^n$ throughout.

(i) [1 mark] Without differentiating, write down the maximum value of $\cos(2x + 30°)$.

(ii) [4 marks] Again without differentiating, find the maximum value of

$$\cos(2x + 30°)\,(1 - \cos(2x + 30°))\,.$$

(iii) [4 marks] Hence write down the maximum value of

$$\cos^5(2x + 30°)\,(1 - \cos(2x + 30°))^5\,.$$

(iv) [6 marks] Find the maximum value of

$$\left(1 - \cos^2(3x - 60°)\right)^4 (3 - \cos(150° - 3x))^8\,.$$

Solution

(i) 1

(ii) Write $f(y) = y(1-y)$, where $y = \cos(2x + 30°)$

$f(y)$ is an n-shaped quadratic, with roots $y = 0$ & 1, and its maximum occurs when $y = \frac{1}{2}(0+1) = \frac{1}{2}$ and

$f\left(\frac{1}{2}\right) = \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}$ (noting that $\cos(2x + 30°) = \frac{1}{2}$ has a solution).

(iii) Write $g(y) = y^5(1-y)^5 = [f(y)]^5$

As $y = x^5$ is an increasing function, $g(y)$ is maximised when $f(y)$ is maximised. So the maximium value of $g(y)$ is $(\frac{1}{4})^5 = \frac{1}{1024}$

[Quite a generous 4 marks.]

(iv) $(1 - cos^2(3x - 60°))^4(3 - \cos(150° - 3x))^8$

$= (sin^2(3x - 60))^4(3 - \sin(90° - [150° - 3x]))^8$

$= sin^8(3x - 60°)(3 - \sin(3x - 60°))^8$

The maximum value of $h(y) = y(3-y)$ occurs when $y = \frac{3}{2}$.

However, there is no solution to $\sin(3x - 60°) = \frac{3}{2}$.

The largest possible value for $h(y)$ when $y = \sin(3x - 60°)$ is $1(3-1) = 2$, and the smallest value occurs when y is as negative as possible; ie when $y = -1$ and $h(y) = (-1)(3 - [-1]) = -4$.

And so the maximum value of the given expression is $(-4)^8$ or 2^{16}.