2023 MAT – Q2 (4 pages; 17/8/25)

2. For **ALL APPLICANTS.**

For n a positive whole number, and for $x \neq 0$, let $p_n(x) = x^n + x^{-n}$.

(i) [3 marks] Sketch the graph of $y = p_1(x)$. Label any turning points on your sketch.

(ii) [1 mark] Show that $p_2(x) = p_1(x)^2 - 2$.

(iii) [1 mark] Find an expression for $p_3(x)$ in terms of $p_1(x)$.

(iv) [5 marks] Find all real solutions x to the equation

$$x^4 + x^3 - 10x^2 + x + 1 = 0.$$

(v) [5 marks] Find all real solutions x to the equation

$$x^7 + 2x^6 - 5x^5 - 7x^4 + 7x^3 + 5x^2 - 2x - 1 = 0.$$

Solution

(i) $p_1(x) = x + \frac{1}{x} = \frac{x^2+1}{x}$; so vertical asymptote at $x = 0$

(and $p_1(\delta) < 0$ for $\delta < 0$; $p_1(\delta) > 0$ for $\delta > 0$)

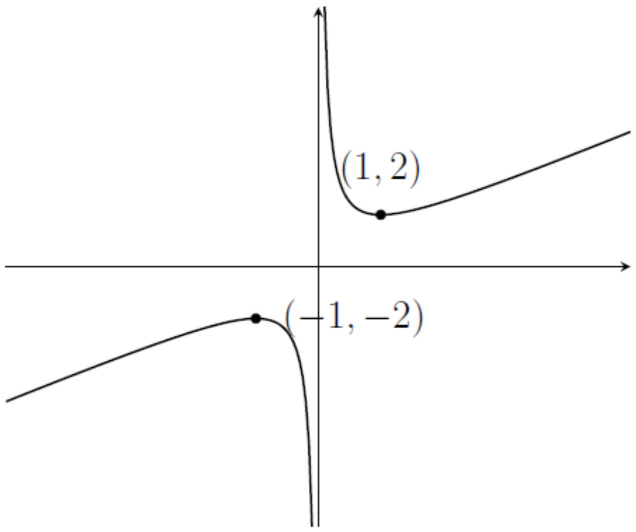
Also $p_1(x) \to \pm\infty$ as $x \to \pm\infty$

$p'_1(x) = 1 - \frac{1}{x^2}$; $p''_1(x) = \frac{2}{x^3}$

$p'_1(x) = 0$ when $x = \pm 1$

$p''_1(1) > 0$, so minimum at $(1,2)$

$p''_1(-1) < 0$, so maximum at $(-1,-2)$



(ii) $[p_1(x)]^2 - 2 = \left(x + \frac{1}{x}\right)^2 - 2 = x^2 + x^{-2} = p_2(x)$

(iii) $p_3(x) = x^3 + x^{-3} = \left(x + \frac{1}{x}\right)^3 - 3x^2.\frac{1}{x} - 3x\,.\frac{1}{x^2}$

$= [p_1(x)]^3 - 3(x + x^{-1)} = [p_1(x)]^3 - 3p_1(x)$

(iv) $x^4 + x^3 - 10x^2 + x + 1 = 0$ (*)

$\Leftrightarrow x^2 + x - 10 + \frac{1}{x} + \frac{1}{x^2} = 0$, on division by x (as $x = 0$ isn't a root of (*));

ie $p_2(x) + p_1(x) - 10 = 0$;

or $[p_1(x)]^2 - 2 + p_1(x) - 10 = 0$, from (ii),

or $[p_1(x)]^2 + p_1(x) - 12 = 0$, (B)

or $(p_1(x) + 4)(p_1(x) - 3) = 0$;

giving $p_1(x) = -4$ *or* 3;

ie $x + \frac{1}{x} = -4$ *or* 3,

so that either $x^2 + 4x + 1 = 0$ or $x^2 - 3x + 1 = 0$; (A)

and hence either $x = \frac{-4\pm\sqrt{12}}{2}$ or $x = \frac{3\pm\sqrt{5}}{2}$;

ie $x = -2 \pm \sqrt{3}$ or $x = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$

[Strictly speaking, we have only shown so far that the roots of the original equation must belong to the set containing the above 4 roots. Conceivably some of the roots may be repeated, with some of the 4 values above not actually satisfying the original equation.]

We can see also that the process is reversible: the above 4 values satisfy (A) and (B) and (*) in turn, so that these values are the roots of the original equation.

(v) [Dividing by some power of x straightaway doesn't seem to lead anywhere, but we can see that 1 is a root of the equation.]

As $x = 1$ can be seen to be a root,

$x^7 + 2x^6 - 5x^5 - 7x^4 + 7x^3 + 5x^2 - 2x - 1 = 0$

$\Leftrightarrow (x-1)(x^6+3x^5-2x^4-9x^3-2x^2+3x+1)=0$

As $x=0$ isn't a root, we can divide the 6th order polynomial by x^3, to give $x^3+3x^2-2x-9-\frac{2}{x}+\frac{3}{x^2}+\frac{1}{x^3}=0$;

ie $p_3(x)+3p_2(x)-2p_1(x)-9=0$,

which can be written as

$\{[p_1(x)]^3-3p_1(x)\}+3([p_1(x)]^2-2)-2p_1(x)-9=0$

or $[p_1(x)]^3+3[p_1(x)]^2-5p_1(x)-15=0$,

a root of which can be seen to be $p_1(x)=-3$ [considering the factors of -15],

so that $(p_1(x)+3)([p_1(x)]^2-5)=0$,

and so $p_1(x)=-3$ or $\pm\sqrt{5}$

When $p_1(x)=-3$, $x+\frac{1}{x}=-3$,

so that $x^2+3x+1=0$,

and $x=\frac{-3\pm\sqrt{5}}{2}$

When $p_1(x)=\pm\sqrt{5}$, $x+\frac{1}{x}=\pm\sqrt{5}$,

so that $x^2\mp\sqrt{5}x+1=0$,

and $x=\frac{\pm\sqrt{5}\pm\sqrt{1}}{2}$

The argument above is reversible again, and so the 7 roots of the original equation are:

$1, -\frac{3}{2}\pm\frac{\sqrt{5}}{2}, \pm\frac{1}{2}\pm\frac{\sqrt{5}}{2}$