2019 MAT - Multiple Choice (9 pages; 27/8/20)

Q1/A

Solution

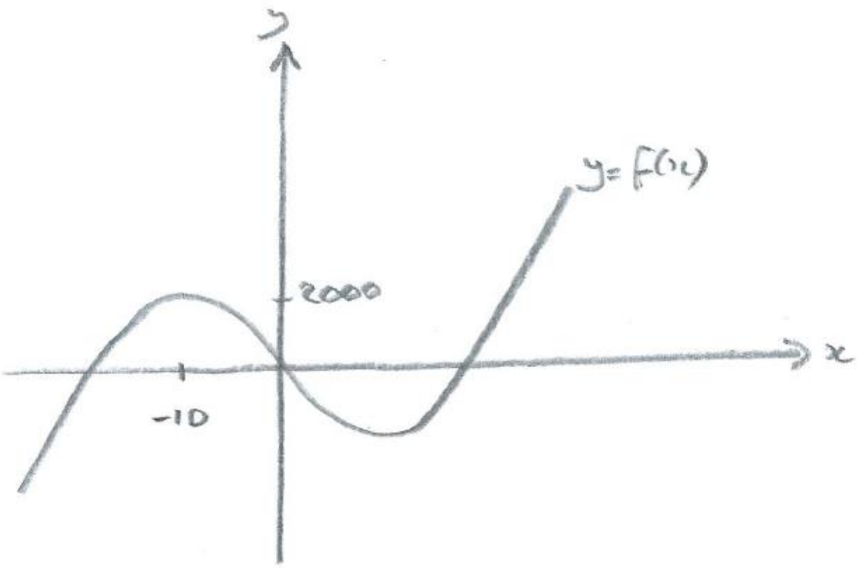
$x^3 - 300x = 3000 \Rightarrow x(x^2 - 300) - 3000 = 0$

Consider $f(x) = x(x^2 - 300)$

$y = f(x)$ crosses the x-axis at $x = 0$ & $\pm\sqrt{300}$

Also $f'(x) = 3x^2 - 300$, so that $f'(x) = 0$ when $x = \pm 10$

And $f(-10) = -10(100 - 300) = 2000$



So the local maximum of $y = f(x)$ is at $(-10, 2000)$ and the

local maximum of $y = f(x) - 3000$ is therefore at $(-10, -1000)$, and so the graph of $y = f(x) - 3000$ crosses the x-axis just once. So there is exactly one real sol'n to $x^3 - 300x = 3000$,

and **the answer is (b).**

Q1/B

Solution

Let the two numbers be a^2 and b^3.

Then $a^2b^3 = (ab)^2b$, and b need not be a square number, in which case $(ab)^2b$ is not a square number, so that (a) is not true.

But b could be a square number, so that $(ab)^2b$ is a square number, and therefore (b) is not true.

If b is a square number, then $a^2b^3 = (ab)^2b$ is a square number.

Suppose that $a = c^3$. Then $a^2b^3 = c^6b^3 = (c^2b)^3$, so that (c) is true.

[As seen above, $(ab)^2b$ could be a square number, and so (d) is not true. And $(ab)^2b$ need not be a square number, so that (e) is not true.]

So the answer is (c).

Q1/C

Solution

Let $f(x) = sin^2x + sin^4x + sin^6x + sin^8x + \cdots$

As $f(0) = 0$, (a) can be ruled out.

As $f(90) = 1 + 1 + 1 + 1 + \cdots$, (b) & (c) can be ruled out.

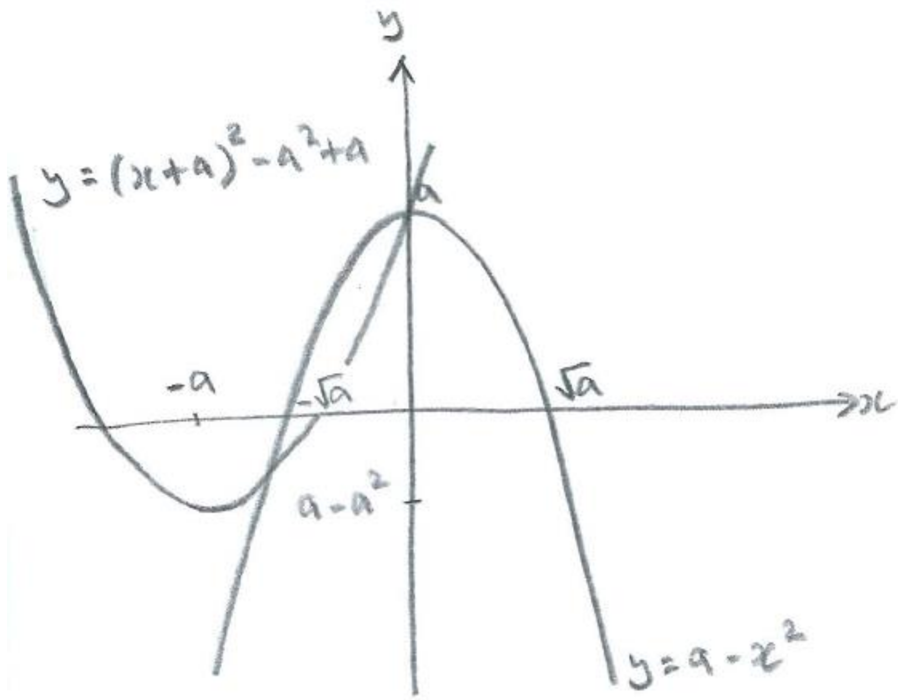
As $f(x) \geq 0$, (e) can be ruled out.

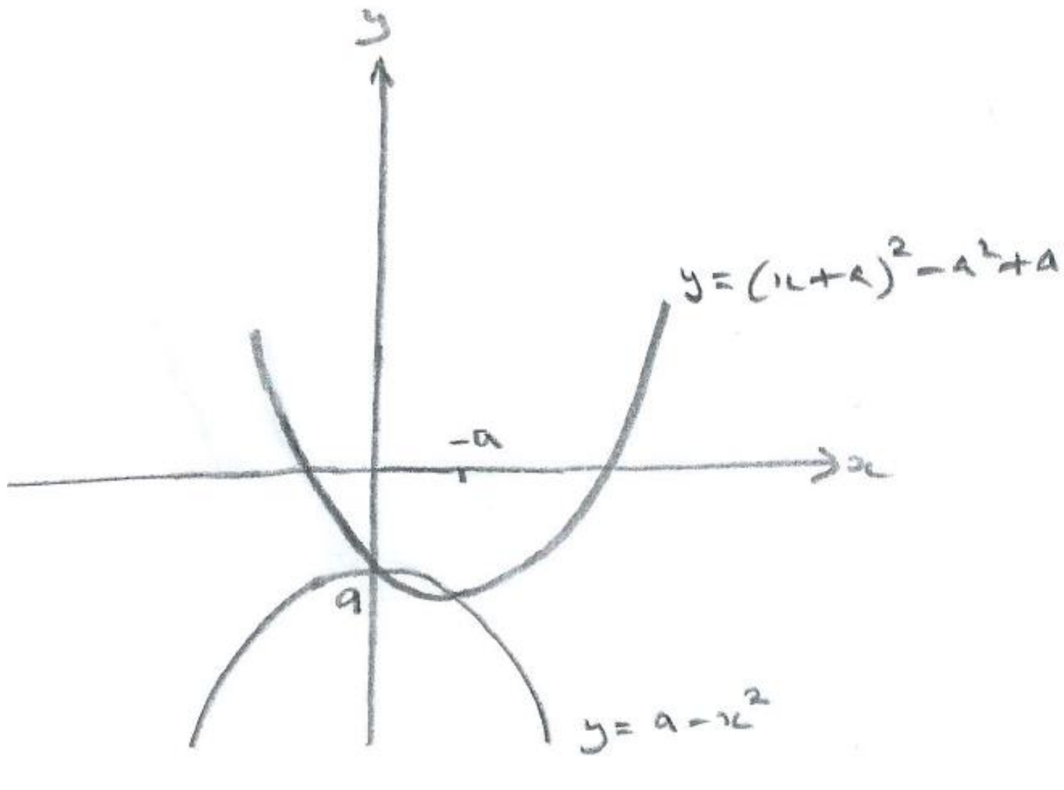
So the answer is (d).

Q1/D

Solution

$x^2 + 2ax + a = (x + a)^2 - a^2 + a$





The diagrams show (provisionally) the cases (A) $a > 0$ and (B) $a < 0$. ((A) is based on $a > 1$, but this doesn't affect the method).

In both cases, the curves intersect when

$x^2 + 2ax + a = a - x^2$;

ie when $2x^2 + 2ax = 0$;

ie $x = 0 \; or - a$

Thus, for (A), the lefthand point of intersection should be at the minimum of $y = (x + a)^2 - a^2 + a$

For (A), the required area = $\int_{-a}^{0} (a - x^2) - (x^2 + 2ax + a) dx$

$= \int_{-a}^{0} -2x^2 - 2ax \, dx$

$= [-\frac{2}{3}x^3 - ax^2]_{-a}^{0}$

$= -(-\frac{2}{3}(-a^3) - a^3)$

$= \frac{1}{3}a^3$

Thus $\frac{1}{3}a^3 = 9$, and so $a = 3$.

For (B), the required area = $\int_{0}^{-a} (a - x^2) - (x^2 + 2ax + a) dx$

$= -\frac{1}{3}a^3$, from the above working

Thus $-\frac{1}{3}a^3 = 9$, and so $a = -3$.

So the answer is (b).

Q1/E

Solution

$siny - sinx = cos^2x - cos^2y$

$\Rightarrow siny - sinx = (1 - sin^2x) - (1 - sin^2y)$

$= sin^2y - sin^2x$

$= (siny - sinx)(siny + sinx)$

So either $siny - sinx = 0$ (A) or $1 = siny + sinx$ (B)

(A) $\Rightarrow y = x + 360k$ or $y = \pi - x + 360k$

ie the sol'n includes an infinite number of straight lines.

So the answer is (e).

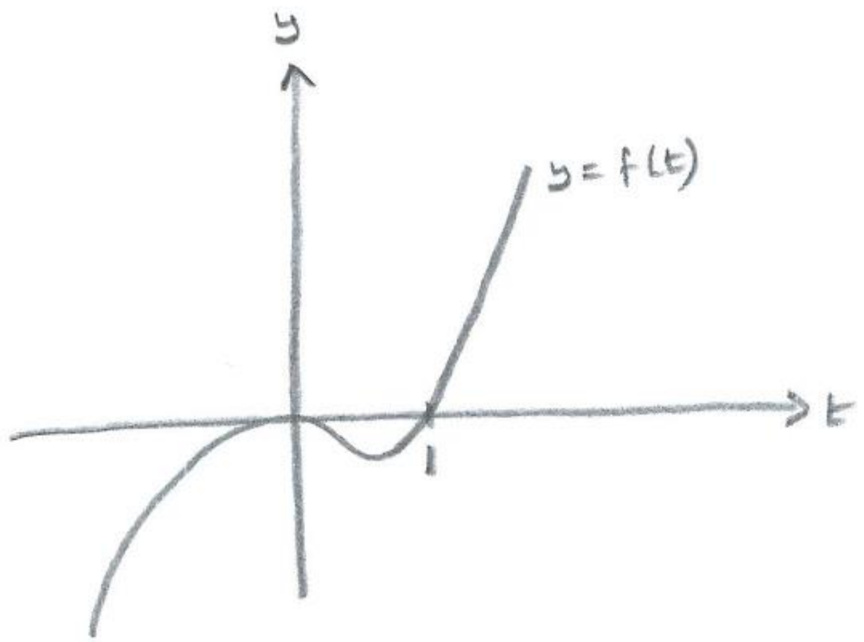
Q1/F

Solution

$sin^3x + cos^2x = 0$

$\Rightarrow sin^3x + 1 - sin^2x = 0$

Let $f(t) = t^3 - t^2 = t^2(t - 1)$ (see diagram)



$f'(t) = 3t^2 - 2t$

$f'(t) = 0 \Rightarrow t = 0 \; or \; \frac{2}{3}$

and $f\left(\frac{2}{3}\right) = \frac{4}{9}\left(-\frac{1}{3}\right) = -\frac{4}{27}$

As $\frac{4}{27} < 1$, $y = f(t) + 1$ crosses the t-axis once.

Also $f(-1) + 1 = 1(-2) + 1 = -1$, and $f(0) + 1 = 1$,

so that there is one sol'n of $t^3 - t^2 + 1 = 0$

with $t \in (-1,0)$

Hence, for $0° \leq x < 360°$, there are 2 sol'ns for x.

So the answer is (c).

Q1/G

Solution

$log_b a = \frac{1}{log_a b}$, so that $\frac{1}{c} = c + \frac{3}{2}$

$\Rightarrow 2c^2 + 3c - 2 = 0$

$\Rightarrow c = \frac{-3 \pm \sqrt{9+16}}{4}$

$\Rightarrow c = \frac{1}{2}$, as $c > 0$

Then $log_b a = 2$ & $log_{\frac{1}{2}} a = b$,

so that $a = b^2$ & $a = \left(\frac{1}{2}\right)^b$

As the graphs of $y = x^2$ & $y = 2^{-x}$ cross once, for $x > 0$,

there are unique values for a & b also.

So the answer is (a).

Q1/H

Solution

Let the sides be $\frac{a}{r}, a \,\&\, ar$ (where $r > 0$)

Then $tan\angle BAC = \frac{a}{\left(\frac{a}{r}\right)} = r$ or $\frac{\left(\frac{a}{r}\right)}{a} = \frac{1}{r}$

By Pythagoras, $\left(\frac{a}{r}\right)^2 + a^2 = (ar)^2$

$\Rightarrow 1 + r^2 = r^4$ or $(r^2)^2 - r^2 - 1 = 0$

$\Rightarrow r^2 = \frac{1\pm\sqrt{5}}{2}$; ie $r = \sqrt{\frac{1+\sqrt{5}}{2}}$, as $r^2 > 0$ and $r > 0$

And $\frac{1}{r} = \sqrt{\frac{2}{1+\sqrt{5}}} = \sqrt{\frac{2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})}} = \sqrt{\frac{2(1-\sqrt{5})}{1-5}} = \sqrt{\frac{\sqrt{5}-1}{2}}$

So the answer is (c).

Q1/I

Solution

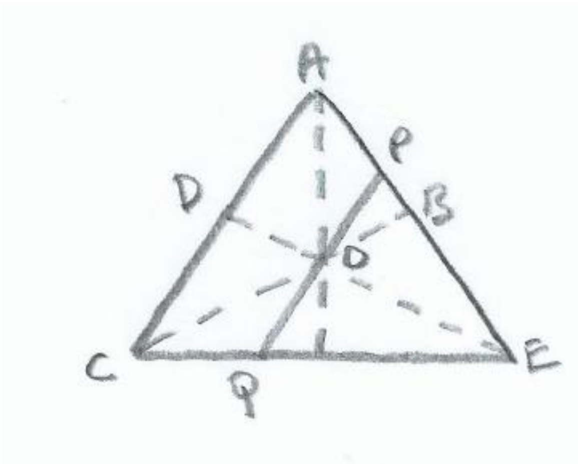
$y = x2^x$ is a strictly increasing function, as both $y = x$ and $y = 2^x$ are strictly increasing.

So if $x2^x = y2^y$, then $x = y$.

So the answer is (a).

Q1/J

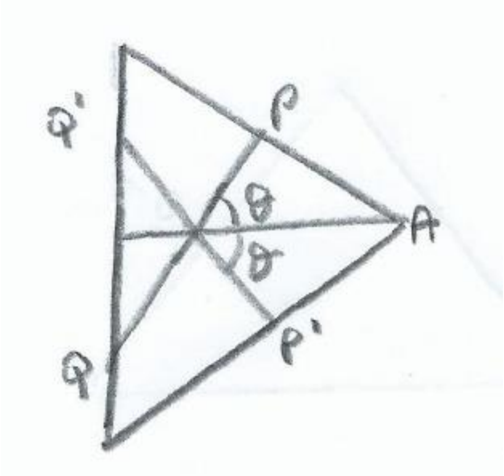
Solution



Without loss of generality, P can be taken to lie between A and B. By symmetry, PQ will be parallel to AC (so that $PA = QC$). O lies $\frac{2}{3}$ of the way from E to D [standard result], and so PQ has $\frac{2}{3}$ of the length of AC, by similar triangles; ie $PQ = \frac{2}{3}$.

So the answer is (d).

[The official sol'n is presumably referring to an equilateral triangle drawn as below. $P'Q'$ makes an angle $-\theta$ with the positive x-axis. The period of 60° can be seen from the 1st diagram: PQ coincides with a median of the triangle every 60°, and by symmetry $f(\theta)$ repeats itself.]


Q'
P
θ
θ
A
P'
Q