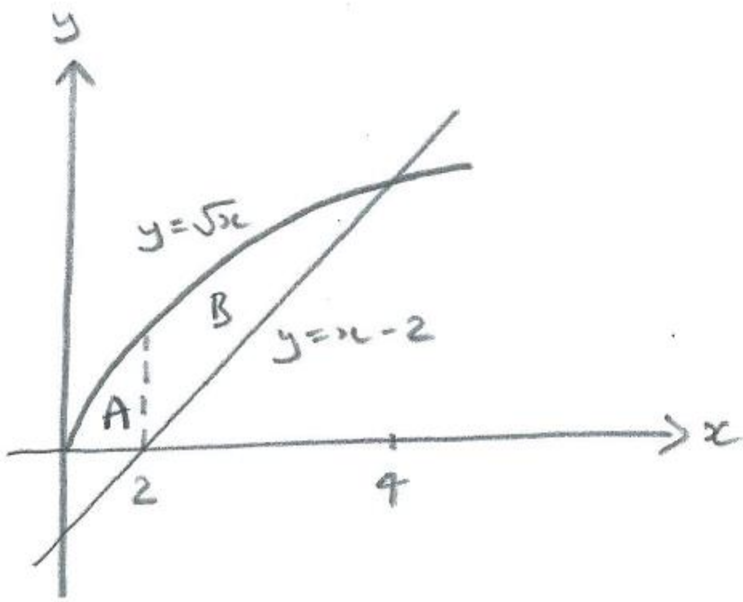
2018 MAT - Multiple Choice (8 pages; 27/8/20)

Q1/A

Solution

Note that $\sqrt{x}$ means the positive root (consider the quadratic formula, which involves $\pm\sqrt{\ }$).



$y = \sqrt{x}$ meets $y = x - 2$ when $\sqrt{x} = x - 2 \Rightarrow x = (x-2)^2$

$\Rightarrow x^2 - 5x + 4 = 0 \Rightarrow (x-4)(x-1) = 0 \Rightarrow x = 4$

($x = 1$ arises from $-\sqrt{x} = x - 2$)

A+B $= \int_0^2 x^{\frac{1}{2}}\,dx + \int_2^4 x^{\frac{1}{2}} - (x-2)dx$

$= \int_0^4 x^{\frac{1}{2}}\,dx - \int_2^4 x - 2\,dx$

$= \left[\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^4 - \left[\frac{1}{2}x^2 - 2x\right]_2^4$

$= \frac{2}{3}(8-0) - (0-[-2])$

$= \frac{16}{3} - 2 = \frac{10}{3}$

$\Rightarrow$ **Answer is (d)**

Q1/B

Solution

Substituting $y = e^{kx}$ into the equation,

$(k^2e^{kx} + ke^{kx})(ke^{kx} - e^{kx}) = e^{kx}.ke^{kx}$

$\Rightarrow (k^2 + k)(k-1) = k,$ as $e^{kx} \neq 0$

$\Rightarrow k = 0$ or $(k+1)(k-1) = 1$

$\Rightarrow k = 0$ or $k^2 = 2$

So there are 3 distinct values of k that satisfy the equation;

$\Rightarrow$ **Answer is (d)**

Q1/C

Solution

$ax^2 + c = bx^2 + d \Rightarrow (a-b)x^2 = d - c$

$\Rightarrow x^2 = \frac{d-c}{a-b}$, provided that $a \neq b$

If $a = b$, then the curves either don't meet at all, or are the same curve (when $c = d$). So $a = b$ can be discounted.

There are then two distinct values of x when $a - b$ and $d - c$ are either both positive or both negative.

$\Rightarrow$ **Answer is (e)**

Q1/D

Solution

$y = f(x-2)$ is obtained from $y = f(x)$ by a translation of 2 units to the right

$x^2 - 5x + 7 = \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + 7$

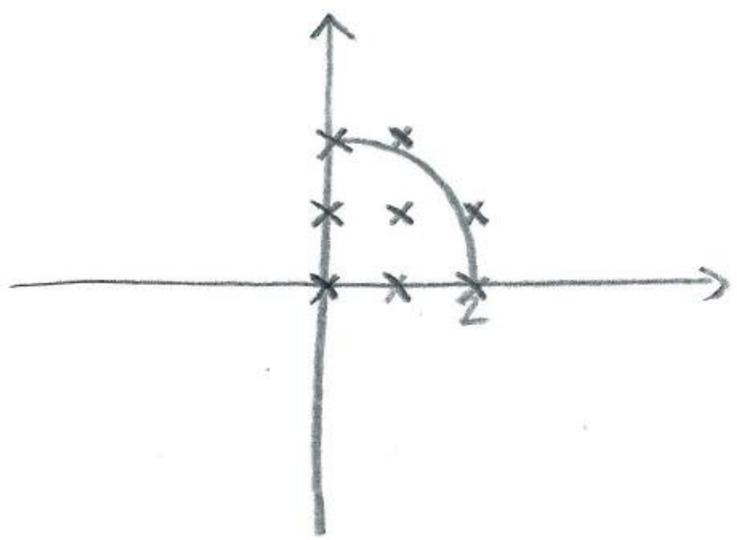
$\Rightarrow$ minimum at $(\frac{5}{2}, \frac{3}{4})$

So minimum of $y = f(x-2)$ is at $(\frac{9}{2}, \frac{3}{4})$

$\Rightarrow$ **Answer is (b)**

Q1/E

Solution



Referring to the diagram, there will be 13 grid points within or on the circle of radius 2 (1 at the Origin, 4 more on the x-axis, 4 more on the y-axis, and 1 other in each of the 4 quadrants).

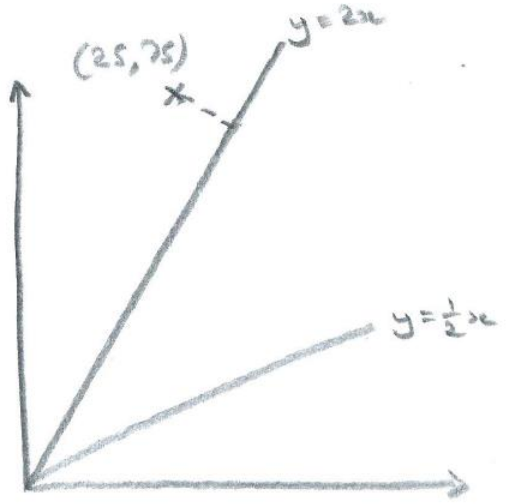
Then $2n - 5 = 21$, so that 8 further grid points are needed, and this occurs when the radius is increased to $\sqrt{1^2 + 2^2} = \sqrt{5}$

⇒ **Answer is (a)**

Q1/F

Solution

[This is a good example of a question where a diagram reveals a hidden feature of the problem.]



The points that the particle can reach are of the form

$p\binom{2}{1} + q\binom{1}{2}$

The two extreme possibilities are where $p = 0$ and $q = 0$.

These are represented by the lines $y = 2x$ and $y = \frac{1}{2}x$, respectively.

Thus the possible points lie within these two lines (which have gradients of $\frac{1}{2}$ and 2).

We now note that the point $(25, 75)$ lies outside this region, as the line joining this point to the Origin has gradient 3.

So (a) can be eliminated.

Were non-integer values of q allowed, it would just be a matter of finding the point on the line $y = 2x$ that is closest to $(25, 75)$.

We can however investigate $p = 0$:

The general distance of the particle from $(25, 75)$ is

$d = \sqrt{(25 - 2p - q)^2 + (75 - p - 2q)^2}$,

and when $p = 0, d = \sqrt{(25 - q)^2 + (75 - 2q)^2}$

and this is minimised when

$5q^2 - 350q + 25^2 + (3 \times 25)^2$ is minimised;

ie when $q^2 - 70q + 1250 = (q - 35)^2 - 1225 + 1250$ is minimised;

ie when $q = 35$, and $d = \sqrt{10^2 + 5^2} = 5\sqrt{5}$

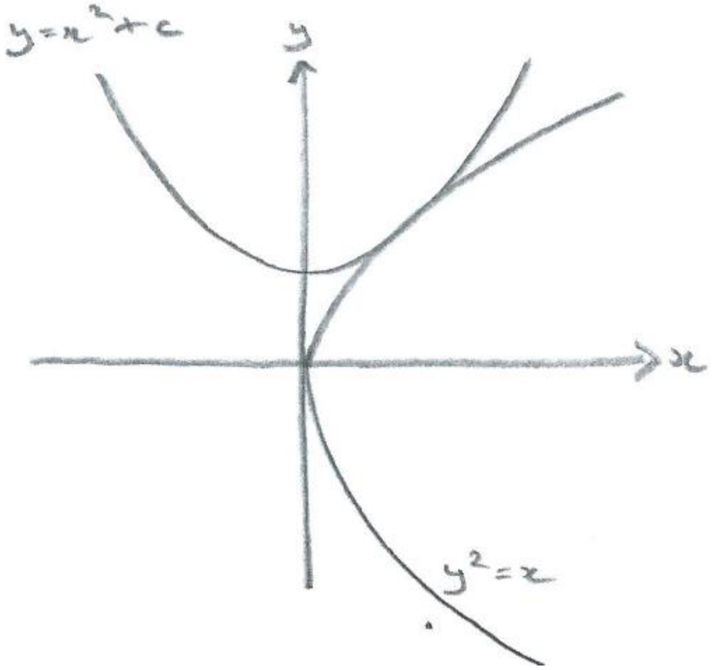
As this in fact occurs for an integer value of q, we have shown that the shortest distance is $5\sqrt{5}$.

⇒ **Answer is (b)**

[The official sol'n says that "we want the closest point on the boundary of the wedge", but had the value of q not been an integer, it might have been the case that a grid point within the wedge was the one nearest to $(25, 75)$.]

Q1/G

Solution



Referring to the diagram, the two curves will only touch tangentially when $c > 0$.

The gradients of the two curves are equal when

$2x = \frac{1}{2}x^{-\frac{1}{2}}$

$\Rightarrow 16x^2 = 1/x$

$\Rightarrow x = 16^{-\frac{1}{3}}$ and $y = \left(16^{-\frac{1}{3}}\right)^{\frac{1}{2}} = 16^{-\frac{1}{6}}$

Then $y = x^2 + c \Rightarrow c = 16^{-\frac{1}{6}} - 16^{-\frac{2}{3}}$

$= 16^{-\frac{1}{6}}(1 - 16^{-\frac{3}{6}})$

$= 16^{-\frac{1}{6}}(1 - \frac{1}{4})$

$= \frac{3}{4} . 4^{-\frac{1}{3}}$

$\Rightarrow$ **Answer is (b)**

Q1/H

Solution

[This is a bit of a trick question - the triangles inside the circle turning out to be a red herring.]

[We could try to derive some relation involving the areas of the triangles, and hope that it gives rise to the expression $\frac{4s^2+t^2}{5st}$, but this would be time-consuming and highly risky. Instead we can quickly see what happens if we rearrange the given expression.]

$E = \frac{4s^2+t^2}{5st} = \frac{4s}{5t} + \frac{t}{5s} = \frac{4}{5}r + \frac{1}{5}r^{-1}$, writing $r = \frac{s}{t}$

[As the smallest value is required, we could quickly see what $\frac{dE}{dr} = 0$ leads to.]

Then $\frac{dE}{dr} = 0 \Rightarrow \frac{4}{5} - \frac{1}{5}r^{-2} = 0 \Rightarrow r^{-2} = 4 \Rightarrow r = \frac{1}{2}$

and when $r = \frac{1}{2}$, $E = \frac{4}{5}\left(\frac{1}{2}\right) + \frac{1}{5}(2) = \frac{4}{5}$

This means that, whatever values T and S happen to have, E cannot be lower than $\frac{4}{5}$. The question is: does the geometrical set-up place any further constraint on E?

But it is clear that we can construct an example where $s = \frac{1}{2}t$, and so $\frac{4}{5}$ is the minimum value of $\frac{4s^2+t^2}{5st}$.

ie the answer is (c)

Q1/I

Solution

When $x = 0, y^8 = 1 \Rightarrow y = \pm 1$

So only (c) or (d) are possible.

Only (d) has symmetry about the x-axis, which means that replacing y with $-y$ leads to the same value(s) of x.

But this isn't true for the curve in question.

So, by elimination, **(c) has to be correct**.

Q1/J

Solution

D shows $y = -x$ or $x + y = 0$, so D is possible, and we can therefore eliminate (b) and (e).

C shows $x^2 + y^2 = r^2$ or $x^2 - \frac{1}{2}r^2 + y^2 - \frac{1}{2}r^2 = 0$, so C is possible, and we can therefore eliminate (a).

This leaves (c) and (d). So the key question is whether A is possible.

Given that only polynomials are involved, A must show $y = x^2 - c$ (or possibly $y = x^4 - c$), and this isn't of the required form.

So A is not possible, and **the answer must be (c)**.