2017 MAT paper - Q2 (2 pages; 28/8/20)

Solution

(i) Let $f(x) = x^3 + x^2 - 1$

$f(0) = -1$ & $f(1) = 1$

As $f(x)$ is a continuous function, the change of sign means that a root exists between 0 and 1; ie $0 < \alpha < 1$.

(ii) $\alpha^3 + \alpha^2 = 1 \Rightarrow \alpha^4 + \alpha^3 = \alpha$

$\Rightarrow \alpha^4 = \alpha - \alpha^3 = \alpha - (1 - \alpha^2) = -1 + \alpha + \alpha^2$

(iii)(a) $\alpha^3 + \alpha^2 = 1 \Rightarrow \alpha^2 + \alpha = \alpha^{-1}$

(b) $1 - \alpha + \alpha^2 - \alpha^3 + \alpha^4 - \alpha^5 + \cdots = \frac{1}{1-(-\alpha)}$

(being a GP with common ratio $-\alpha$)

Then $\alpha^3 + \alpha^2 = 1 \Rightarrow \alpha^2(\alpha + 1) = 1 \Rightarrow \frac{1}{1+\alpha} = \alpha^2$

Thus $1 - \alpha + \alpha^2 - \alpha^3 + \alpha^4 - \alpha^5 + \cdots = \alpha^2$

(c) $(1 - \alpha)^{-1} = 1 + \alpha + \alpha^2 + \alpha^3 + \cdots$ (A)

From (b), $1 - \alpha + \alpha^2 - \alpha^3 + \alpha^4 - \alpha^5 + \cdots = \alpha^2$ (B)

Then, adding (A) & (B):

$2(1 + \alpha^2 + \alpha^4 + \cdots) = (1 - \alpha)^{-1} + \alpha^2$ (C)

And $1 + \alpha^2 + \alpha^4 + \cdots = \frac{1}{1-\alpha^2} = \frac{1}{\alpha^3}$ (from $\alpha^3 + \alpha^2 = 1$)

So (C) $\Rightarrow (1-\alpha)^{-1} = \frac{2}{\alpha^3} - \alpha^2$

$= 2(\alpha^2+\alpha)^3 - \alpha^2$, from (a)