2016 MAT - Multiple Choice (8 pages; 27/8/20)

Q1/A

Solution

$a_r = l^{r-1}$

$\prod_{r=1}^{15} a_r = 1.l.l^2 \ldots .l^{14} = l^{\frac{1}{2}(14)(15)} = l^{105}$

So the answer is (d).

Q1/B

Solution

Let the hexagon have sides of length x.

Then $x + \frac{x}{\sqrt{2}} = 1$,

so that $x = \frac{1}{1+\left(\frac{1}{\sqrt{2}}\right)} = \frac{\sqrt{2}}{\sqrt{2}+1} = \sqrt{2}\left(\sqrt{2}-1\right) = 2 - \sqrt{2}$

So the answer is (b).

Q1/C

Solution

$x^2 + ax + y^2 + by = c$

$\Rightarrow (x + \frac{a}{2})^2 + (y + \frac{b}{2})^2 - \frac{a^2}{4} - \frac{b^2}{4} = c$

So the centre is $(-\frac{a}{2}, \frac{b}{2})$ and the radius $= \sqrt{c + \frac{a^2}{4} + \frac{b^2}{4}}$

The Origin will lie within the circle when the distance of the centre from the Origin is less than the radius;

ie when $\sqrt{(\frac{a}{2})^2 + (\frac{b}{2})^2} < \sqrt{c + \frac{a^2}{4} + \frac{b^2}{4}}$

and $(\frac{a}{2})^2 + (\frac{b}{2})^2 < c + \frac{a^2}{4} + \frac{b^2}{4}$;

ie $c > 0$

So the answer is (a).

Q1/D

Solution

$cos^n(x) + cos^{2n}(x) = 0$

$\Rightarrow cos^n(x)\left(1 + cos^n(x)\right) = 0$

$\Rightarrow cosx = 0$ or $cos^n(x) = -1$

If n is even, then there are no sol'ns to $cos^n(x) = -1$, and 2 sol'ns to $cosx = 0$ in the given range.

[So the answer can only be (d).]

If n is odd, then $cos^n(x) = -1 \Rightarrow cosx = -1$, for which there is 1 sol'n in the given range, and there are 2 sol'ns to $cosx = 0$.

Thus, if n is even, then there are 2 sol'ns, and if n is odd, then there are 3 sol'ns.

So the answer is (d).

Q1/E

Solution

Let $f(x) = (x-1)^2 - cos(\pi x)$

$f(0) = 0$; ie the curve should pass through the Origin

So the answer can only be (a), (b) or (c).

The line of symmetry of $y = f(x)$ is $x = 1$

[$f(1+a) = a^2 - cos(\pi + \pi a)$

$= a^2 - cos\pi \cos(\pi a) + sin\pi \sin(\pi a)$

$= a^2 - cos\pi \cos(\pi a)$

$f(1-a) = a^2 - cos(\pi - \pi a)$

$= a^2 - cos\pi \cos(\pi a) - sin\pi \sin(\pi a)$

$= a^2 - cos\pi \cos(\pi a)$,

so that $f(1-a) = f(1+a)$]

and $f(1) = 1$,

so that the answer is (a).

Q1/F

Solution

Let $y = x^2$

Then equivalent requirement is that $y + 1$ is a factor of

$f(y) = (3 + y^2)^n - (y+3)^n(y-1)^n$

This occurs when $f(-1) = 0$;

ie when $4^n - 2^n(-2)^n = 0$ (1)

When n is even, (1) becomes $4^n - 2^n 2^n = 0$, which is always true.

When n is odd, (1) becomes $4^n + 2^n 2^n = 0$, which is never true.

So the answer is (b).

Q1/G

Solution

The sequence is 1, 1, 2, 4, 8, 16, ...

Note that eg $16 = (1 + 1 + 2 + 4) + 8 = 8 + 8$; ie the sequence doubles at each step.

So $x_k = 2^{k-1}$ for $k > 0$

and $\sum_{k=0}^{\infty} \frac{1}{x_k} = 1 + \sum_{k=1}^{\infty} 2^{-(1-k)}$

$= 1 + \frac{1}{1-\frac{1}{2}} = 1 + 2 = 3$

So the answer is (d).

Q1/H

Solution

$\int_0^{\sqrt{a}} a - x^2\, dx > -\int_0^{\sqrt[4]{a}} x^4 - a\, dx$ (as $g(x)$ lies below the x-axis)

$\Leftrightarrow \left[ax - \frac{1}{3}x^3\right]_0^{\sqrt{a}} + \left[\frac{1}{5}x^5 - ax\right]_0^{\sqrt[4]{a}} > 0$

$\Leftrightarrow a^{\frac{3}{2}} - \frac{1}{3}a^{\frac{3}{2}} + \frac{1}{5}a^{\frac{5}{4}} - a^{\frac{5}{4}} > 0$

$\Leftrightarrow \frac{2}{3}a^{\frac{3}{2}} - \frac{4}{5}a^{\frac{5}{4}} > 0$

$\Leftrightarrow \frac{1}{15}a^{\frac{5}{4}}\left(10a^{\frac{1}{4}} - 12\right) > 0$

$\Leftrightarrow 5a^{\frac{1}{4}} - 6 > 0$ (as $a > 0$)

$\Leftrightarrow a^{\frac{1}{4}} > \frac{6}{5}$

$\Leftrightarrow a > \left(\frac{6}{5}\right)^4$ (as $y = x^4$ is an increasing function, for $x > 0$)

So the answer is (e).

Q1/I

Solution

Write $y^2 = 1 - x^2 - A$, where $A \geq 0$

Let $f(x) = ax + by = ax + b\sqrt{1 - x^2 - A}$ (taking the positive root, to maximise $f(x)$).

Then $f'(x) = a + \frac{b}{2}(1 - x^2 - A)^{-\frac{1}{2}}(-2x)$

and $f'(x) = 0 \Rightarrow a = \frac{bx}{\sqrt{1-x^2-A}}$

$\Rightarrow a^2(1 - x^2 - A) = b^2x^2$

$\Rightarrow x^2(a^2 + b^2) = a^2(1 - A)$

$\Rightarrow x = \frac{a\sqrt{1-A}}{\sqrt{a^2+b^2}}$ (taking the positive root again, to maximise $f(x)$)

And then $y = \sqrt{1 - \frac{a^2(1-A)}{a^2+b^2} - A}$ (taking the positive root, to maximise $f(x)$)

$= \sqrt{\frac{a^2+b^2-a^2(1-A)-A(a^2+b^2)}{a^2+b^2}}$

$= \sqrt{\frac{b^2(1-A)}{a^2+b^2}}$

And so $f(x)$ is maximised by setting $A = 0$,

giving $f(x) = \frac{a^2}{\sqrt{a^2+b^2}} + \frac{b^2}{\sqrt{a^2+b^2}} = \sqrt{a^2+b^2}$

So the answer is (c).

Alternative method 1 (using 2nd year theory)

Suppose that a sol'n exists for which $x^2 + y^2 < 1$; ie so that the point (x, y) lies inside the unit circle centred on the Origin.

Then it would be possible to increase $ax + by$ by increasing either x or y, until (x, y) lies on the circle.

So $x^2 + y^2 = 1$, and we can write $x = cos\theta, y = sin\theta$ for some θ.

Then $ax + by = acos\theta + bsin\theta$ can be written as

$rsin(\theta + \alpha) = rsin\theta cos\alpha + rsin\alpha cos\theta$,

where $a = rsin\alpha$ & $b = rcos\alpha$,

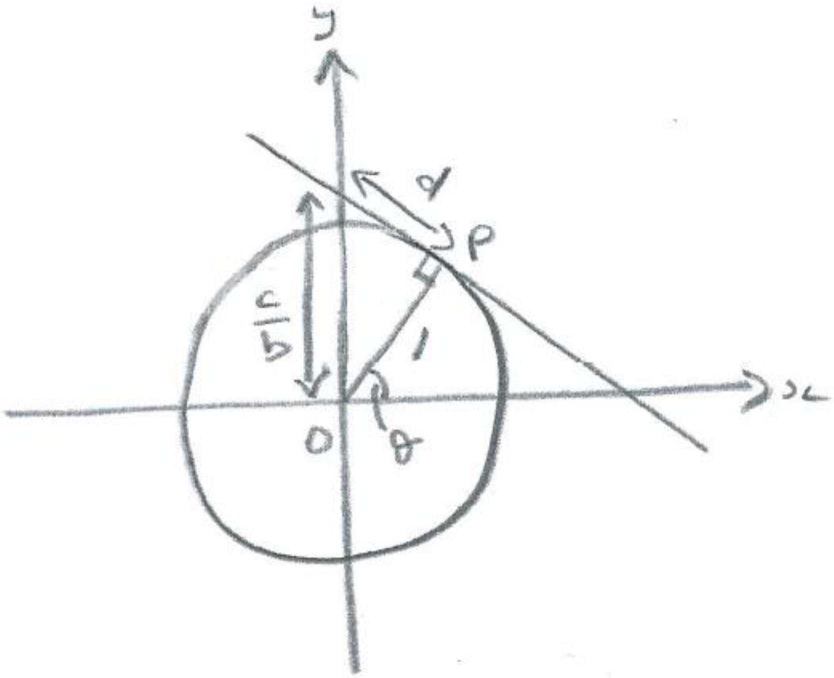
so that $r^2(sin^2\alpha + cos^2\alpha) = a^2 + b^2$, and hence $r = \sqrt{a^2+b^2}$

Thus the largest value that $ax + by$ can equal is $r = \sqrt{a^2+b^2}$.

Alternative method 2

Write $ax + by = c$, and rearrange to $y = -\frac{a}{b}x + \frac{c}{b}$

[The official sol'ns incorrectly says $y = \frac{a}{b}x + \frac{c}{b}$]



This is the family of lines of gradient $-\frac{a}{b}$, and we want the line that crosses the y-axis as high as possible. Referring to the diagram, the gradient of OP is $-\frac{1}{\left(-\frac{a}{b}\right)}=\frac{b}{a}$. Thus $tan\theta=\frac{b}{a}$ and so $\frac{d}{1}=\tan(90-\theta)=\frac{a}{b}$

Then, by Pythagoras' theorem, $1^2+d^2=\left(\frac{c}{b}\right)^2$

so that $b^2+a^2=c^2$,

and $c=\sqrt{a^2+b^2}$

Q1/J

Solution

(a) true; eg $x(n)=4$: this is possible (eg when $x=4$, so that $\Pi(n)=1$), and n cannot be prime

(b) $x(n)$ would have to be odd (if it were 2, then 32 would be an exception to the proposition; other even numbers ⇒ n isn't prime); $3^4 = 81$ rules out $x(n) = 1$; $3^5 = 243$ rules out $x(n) = 3$;

$3^6 = 729$ rules out $x(n) = 9$; $3^7 = 2187$ rules out $x(n) = 7$;

$5^2 = 25$ rules out $x(n) = 5$; so the statement is false

So the answer is (b).

[The official sol'ns give $7^5 = 16807$ as a counterexample for $x(n) = 7$, but it's not an obvious one to try!]

[(c): $x(n) = 0 \Rightarrow$ 2 & 5 must be factors, which contradicts $\Pi(n) = 1$; so (c) is true

(d) If eg $\Pi(n) = 1$ & $x(n) = 1$, in which case n could be eg 11 (prime), or $3^4 = 81$ (not prime); so (d) is true

(e) eg 2×3 gives $x(n) = 6$; 2×5 gives $x(n) = 0$;

2×7 gives $x(n) = 4$; 2×11 gives $x(n) = 2$;

2×19 gives $x(n) = 8$; 3×5 gives $x(n) = 5$;

3×7 gives $x(n) = 1$; 3×11 gives $x(n) = 3$;

3×13 gives $x(n) = 9$; 3×19 gives $x(n) = 7$

so (e) is true]