Notes & Solutions for Q1-5 of the Nov. 2015 MAT Paper

(20 pages; 10/8/20)

(to be read in conjunction with the official solutions)

Q1/A - Introduction

Technically a whole number is an integer, which could be positive, negative or zero. However, the mention of a prime number in IV suggests that only positive whole numbers are being considered, as we don't usually refer to -3 (for example) as a prime number.

This question is arguably a bit devious. From the official solutions it is apparent that you are not expected to establish whether each statement is true or false; you only need to deduce which of (a) - (e) must be the correct answer.

Solution

Having established that the final answer takes the form $4(x+1)^2-3$, we can quickly produce a list of the first few final answers (just considering positive x for the moment):
$1, 13, 33, 61$

This reveals that II, III & IV are false, so that the only possible answer is (e).

Usually it is a good idea to check the other statements independently.

From the formula $4(x+1)^2-3$, statement I is seen to be correct, but V is not so quick to deal with (as far as I can see). It can be investigated as follows:

We want to show that $4(x+1)^2-3$ is not $\equiv 2\ mod\ 3$

or that $4(x+1)^2-3 \equiv 0\ or\ 1\ mod\ 3$

We could see what happens if we create a difference of two squares:

Result to prove: $4(x+1)^2 - 4 \equiv 2 \; or \; 0 \; mod \; 3$

LHS $= 4[(x+1)+1][(x+1)-1] = 4x(x+2)$

Going through the whole number values of x, we see that the cases where $4x(x+2) \not\equiv 0 \; mod \; 3$ are:

$4(2)(4), 4(5)(7), 4(8)(10), \ldots$

ie numbers of the form $4(3n-1)(3n+1)$ for $n \in \mathbb{Z}^+$

Then $4(3n-1)(3n+1) = 4(3n)(3n+1) - 4(3n+1)$

$\equiv -4(3n+1) \; mod \; 3$ [as $4(3n)(3n+1)$ is a multiple of 3]

$\equiv -4$ [as $-4(3n)$ is a multiple of 3]

$\equiv 2 \; mod \; 3$, as required

Answer: (e)

Q1/B - Introduction

We could do sketches for small values of n, to get a feel for the problem (and possibly rule out some of the answers). This might be time-consuming though. Alternatively, an algebraic approach may be possible.

Solution

The number of intersections is the number of distinct roots of

$f(x) = f'(x)$; ie $(x+a)^n - n(x+a)^{n-1} = 0$;

or $(x+a)^{n-1}\{(x+a) - n\} = 0$

Thus there will always be 2 distinct roots: $x = -a$ & $x = n - a$ (as $n \neq 0$).

So the answer is (b), as 2 is an even number (another trick question!)

For some reason the official solutions say that the 2nd root will occur for a positive x, but this won't be the case if $a > n$.

Answer: (b)

Q1/C - Notes

The relation $sinx = cos(\frac{\pi}{2} - x)$ can be used for I & III.

Answer: (c)

Q1/D - Notes

[For the unusual integral $\int_0^1 (xt)^2 dt$, we can assume that x isn't a function of t; otherwise $f(x)$ would only be a constant (ie not a function of x). Also, we wouldn't be able to make any progress , if we didn't know the form of the function. The integrals can just be evaluated in terms of x.]

$$f(x) = \int_0^1 (xt)^2 dt = x^2 \int_0^1 t^2 dt = x^2 [\frac{1}{3} t^3]_0^1 = \frac{1}{3} x^2$$

$$g(x) = \int_0^x t^2 dt = \left[\frac{1}{3} t^3\right]_0^x = \frac{1}{3} x^3$$

$$f\left(g(A)\right) = \frac{1}{3} (\frac{1}{3} A^3)^2 = \frac{1}{27} A^6$$

$$g\left(f(A)\right) = \frac{1}{3} (\frac{1}{3} A^2)^3 = \frac{1}{81} A^6$$

Answer: (b)

Q1/E - Introduction

We can make the substitution $u = 2cos(2x) + 2$, noting the range of u, given the range of x.

Solution

With $u = 2cos(2x) + 2$, $0 \leq x \leq 2\pi \Rightarrow 2(-1) + 2 \leq u \leq 2(1) + 2$

ie $0 \leq u \leq 4$

[Note: The official solutions consider $sinx = 0$, with x restricted to $0 \leq x \leq 2\pi$, but this isn't correct.]

Then $sinu = 0 \Rightarrow u = 0 \; or \; \pi$

$\Rightarrow \cos(2x) = -1 \; or \; \frac{\pi-2}{2} = \frac{\pi}{2} - 1$

Now making the substitution $w = 2x$, $0 \leq w \leq 4\pi$

Referring to the graph of $cosw$,

$cosw = -1$ has 2 solutions (for w), and $cosw = \frac{\pi}{2} - 1$ has 4 solutions; making 6 solutions in total.

As $x = \frac{w}{2}$, there are also 6 solutions for x.

[A variation on the above approach is to say that

$2cos(2x) + 2$ must equal $n\pi$, for suitable integer n

Then, either $n = 0$, with $cos(2x) = -1$,

or $n = 1$, with $cos(2x) = \frac{\pi}{2} - 1$

(no other values of n are consistent with $2cos(2x) + 2$),

as before.]

Answer: (d)

Q1/F - Notes

There doesn't seem to be a quick way of answering this question (the official solutions mention the possibility of 'inspection', but this might be wishful thinking). Doing the calculations for each possibility, one at a time, is potentially very time-consuming (and isn't that quick, even though the answer proves to be (b)). This question definitely needs to be left until last (assuming there isn't an even more unfriendly one still to come).

Answer: (b)

Q1/G - Introduction

The relation $cos\theta = sin(\frac{\pi}{2} - \theta)$ can be useful, as an algebraic way of dealing with $-cos\theta$ (although referring to the graph of $cos\theta$ is quicker).

Solution

$cos^2x = cos^2y \Rightarrow cosx = \pm cosy$

Then $cosx = cosy \Rightarrow y = x + 2n\pi$ or $y = (2\pi - x) + 2n\pi$

ie $y = 2n\pi \pm x$

[Alternatively, the two base angles can be taken as x & $-x$]

And $cosx = -cosy \Rightarrow cosy = -sin\left(\frac{\pi}{2} - x\right) = sin(x - \frac{\pi}{2})$

$= cos\left(\frac{\pi}{2} - \left(x - \frac{\pi}{2}\right)\right) = cos(\pi - x)$

[or just refer to the graph of $cosx$, to see that

$-cosx = cos(\pi - x)$]

Thus $y = \pi - x + 2n\pi = (2n + 1)\pi - x$,

or $y = 2\pi - (\pi - x) + 2n\pi = (2n + 1)\pi + x$

ie $y = (2n+1)\pi \ \pm x$

Thus the possible solutions are:

$y = 2n\pi \ \pm x$ and $y = (2n+1)\pi \ \pm x$;

ie straight lines with gradients of ± 1, with y-intercepts of even and odd multiples of π,

making (c) the correct answer

Answer: (c)

Q1/H - Introduction

A standard option is to rearrange an equation or expression into a more convenient form.

Notes

Rearranging into the form $4 - 5x^2 - 6x^3 = (x^2+2)^2$, we then find that there are two distinct roots.

Answer: (c)

Q1/I - Introduction

As well as noting the points of intersection of the 3 curves, we can also consider their gradients in appropriate regions.

Notes

If $f(x) = x^3, g(x) = x^4$ & $h(x) = x^5$,

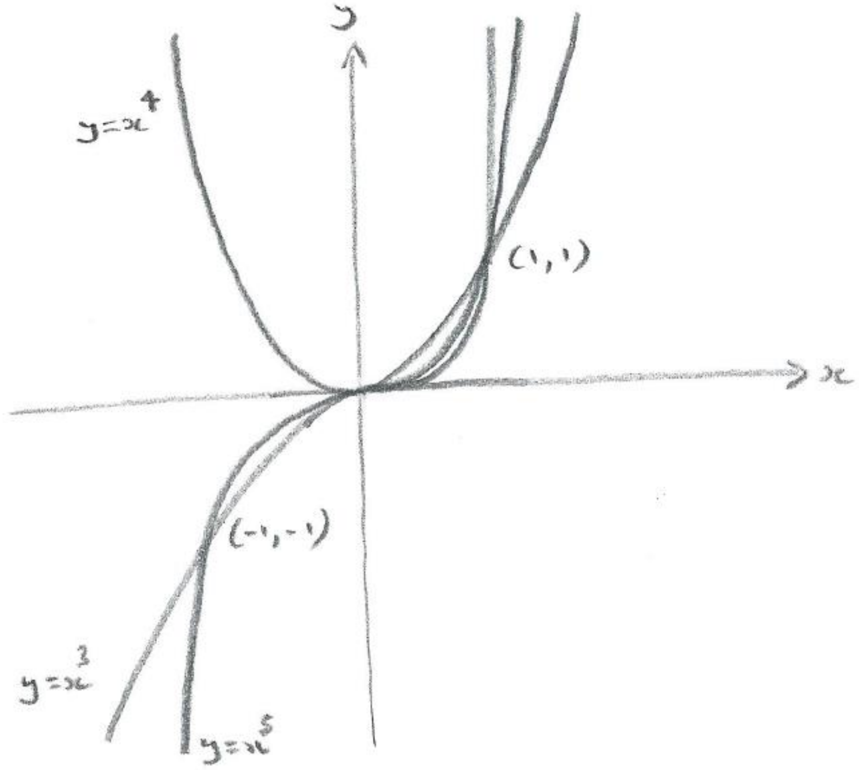
then $f'(x) = 3x^2, g'(x) = 4x^3$ & $h'(x) = 5x^4$,

so that, for $x > 0$ for example, $g'(x) > f'(x) \Rightarrow x > \frac{3}{4}$

and $h'(x) > g'(x) \Rightarrow x > \frac{4}{5}$;

ie $g(x)$ is initially less steep than $f(x)$, but the curves cross at $x = 1$; and similarly for $h(x)$ & $g(x)$

There are 9 regions in the diagram below.



Answer: (d)

Q1/J - Introduction

If we want to compare expressions x & y, we can, for example, compare x^2 & y^2 instead, or see whether $\frac{x}{y} > 1$, or $x - y > 0$. The last option is often easiest. For awkward expressions such as $\frac{log_2 30}{log_3 85}$, it may well be the case that an approximation will do. It's certainly best to leave it until last, when we know which of the other expressions it needs to be compared with (hopefully a simple one, such as $\frac{5}{4}$).

Notes

The official solutions for this type of question often give the impression that the correct answer is already known. In practice, you may need to go down some blind alleys in arriving at the solution.

(a) and (b) are quickest to deal with first: squaring (a), we see that $\frac{7}{4} > \frac{5}{4}$

(e) is probably the next simplest, and we can consider the difference of squares:

$\frac{7}{4} - \frac{(1+2\sqrt{6}+6)}{9} = \frac{63-28-8\sqrt{6}}{36} > \frac{35-8(3)}{36} > 0$

So (a)>(e)

[We could investigate $\frac{\left(\frac{7}{4}\right)}{\left(\frac{7+2\sqrt{6}}{9}\right)} = \frac{63(7-2\sqrt{6})}{4(49-24)} = \frac{63(7-2\sqrt{6})}{100}$, but it isn't as easy to show that this expression is greater than 1 (and we might just end up having to demonstrate an equivalent result of the form $A - B > 0$)]

Next, we can consider $(c)^2 \div (a)^2$ [the expressions lend themselves better to showing that $\frac{B}{A} > 1$]

Thus $\frac{\left(\frac{10!}{9(6!)^2}\right)}{\left(\frac{7}{4}\right)} = \frac{(10)(9)(8)(7)(4)}{(9)(6!)(7)} = \frac{(10)(8)}{(6)(5)(3)(2)} = \frac{8}{18} < 1$, so that (a)>(c).

Finally we need to either compare (d) with (a) directly, or perhaps show that it less than (b) [being simpler than (a)].

Considering the powers of 2 and 3, we see that $log_2 30$ is close to 5, whilst $log_3 85$ is close to 4. [In general, we can only expect to use fairly good approximations to demonstrate inequalities.]

So $\frac{log_2 30}{log_3 85} < \frac{5}{4}$, and $(d) < (b)$. Thus (a) is the largest one.

Answer: (a)

Q2 - Sol'n / Notes

(i) The expression simplifies to $a^{n+1} - b^{n+1}$. This is a standard result, valid for all integer n [viewed as a function of a , $f(a)$ say, $f(b) = b^{n+1} - b^{n+1} = 0$ for all n, so that $(a - b)$ is a factor]. But the companion result

$$a^{n+1} + b^{n+1} = (a + b)(a^n - a^{n-1}b + a^{n-2}b^2 - \cdots - ab^{n-1} + b^n)$$

[note that the signs alternate]

is only valid for even n, since (viewed as a function of a)

$f(-b) = (-b)^{n+1} + b^{n+1} = 0$ (and $(a + b)$ is a factor) only if n is even.

(ii) Suppose that $p = n^2 - 1$, where p is prime.

Then $p = (n - 1)(n + 1)$. As p is prime, we must have

$n - 1 = 1$ & $n + 1 = p$; ie $n = 2$ & $p = 3$

So there are no other prime numbers with this property.

(iii) Let $p = n^3 + 1 = (n + 1)(n^2 - n + 1)$, where p is prime and $n > 0$.

As p is prime and $n > 0$, we must have

$n + 1 = p$ & $n^2 - n + 1 = 1$

Thus $n(n - 1) = 0$, so that $n = 1$ and $p = 2$

ie the only prime number with this property is 2

(iv) The result in (i) can't be applied with $a = 3$ & $b = 2$, as this just gives a factor of $a - b = 1$

However, $3^{2015} - 2^{2015}$ can be arranged as $(3^{403})^5 - (2^{403})^5$

[or as $(3^5)^{403} - (2^5)^{403}$, or even as $(3^{31})^{13\times5} - (2^{31})^{13\times5}$ etc]

so that $3^{403} - 2^{403}$ is a factor.

Thus $3^{2015} - 2^{2015}$ isn't a prime number.

(v) [It is natural to wonder if one of the previous parts is relevant. The official solution manages to use (i) in its alternative approach.]

By way of exploration, we can consider

$(k+1)^3 = k^3 + 3k^2 + 3k + 1 > k^3 + 2k^2 + 2k + 1$ (for $k > 0$)

and so the solution is very simple: the given expression lies between k^3 & $(k+1)^3$, and hence there is no positive integer k for which $k^3 + 2k^2 + 2k + 1$ is a cube.

[Note that there are a couple of errors in the official solution: In the first line it says "Note that $k^3 < k^3 + 2k^2 + 2k$". Presumably "Note that $k^3 < k^3 + 2k^2 + 2k + 1$" was intended.

Then in the 4th line of the alternative approach it says:

"So $n \geq k + 1$, so $n^2 + nk + k^2 \leq 3k^2 + 3k + 1$", where the $\leq$ should be a $\geq$

(Incidentally, this is a good example of why it's important to include your working: Had the statement read:

"So $n \geq k + 1$, so $n^2 + nk + k^2 \leq (k+1)^2 + (k+1)k + k^2 = 3k^2 + 3k + 1$", then the error would have been much clearer, and the rest of the argument might have been considered - or the error may have been spotted by the candidate.)]

Q3 - Sol'n / Notes

(i) [We can of course look ahead in the question, to see what sort of functions might be used.]

$f(x) = x,\ \ g(x) = x + \frac{1}{100}$

(ii) $|f(x) - g(x)| = \frac{1}{400} sin(4x^2)$ when $0 \le x \le \frac{1}{2}$

(as $4\left(\frac{1}{2}\right)^2 = 1 < \pi$, so that $sin(4x^2) > 0$

and $\frac{1}{400} sin(4x^2) \le \frac{1}{400}\sin(1) < \frac{1}{400}\sin\left(\frac{\pi}{3}\right) = \frac{1}{400}\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{320}\left(\frac{320\sqrt{3}}{800}\right) = \frac{1}{320}\left(\frac{4\sqrt{3}}{10}\right)$

$< \frac{1}{320}\left(\frac{4\times1.8}{10}\right) = \frac{1}{320}(0.72) < \frac{1}{320}$

(iii) $g(x) = 1 + \int_0^x 1 + t + \frac{t^2}{2} + \frac{t^3}{6}\ dt$

$= 1 + \left[t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24}\right]_0^x$

$= 1 + \ x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$

Then $|f(x) - g(x)| = \frac{x^4}{24} \le \frac{1}{16(24)} = \frac{1}{384} < \frac{1}{320}$ when $0 \le x \le \frac{1}{2}$

(iv) RHS $= g(x) - f(x) + \int_0^x \big(h(t) - f(t)\big)dt$

$= 1 + \int_0^x f(t)\ dt - f(x) + \int_0^x h(t)dt - \int_0^x f(t)dt$

$= 1 + \int_0^x h(t)dt - f(x) = h(x) - f(x) = \text{LHS}$

(v) [This is a stand-alone result; ie not needing to be derived from the earlier results.]

Consider the area under the graph of $h(t) - f(t)$, between 0 & x.

Assume for the moment that the graph lies above the t-axis.

The maximum height of the function is $h(x_0) - f(x_0)$, and the area under the graph is no greater than the rectangle with base x and height $h(x_0) - f(x_0)$.

As $x \leq \frac{1}{2}$, the rectangle has area $\leq \frac{1}{2}(h(x_0) - f(x_0))$.

As the integral would have a smaller value if part of the graph were to lie below the t-axis,

$\int_0^x \big(h(t) - f(t)\big)dt \leq \frac{1}{2}(h(x_0) - f(x_0))$ whenever $0 \leq x \leq \frac{1}{2}$

(vi) Result to prove: $|f(x) - h(x)| \leq \frac{1}{100}$ for $0 \leq x \leq \frac{1}{2}$

or, as we are told that $f(x) \leq h(x)$,

and if we set $k(x) = h(x) - f(x)$,

result to prove is $k(x) \leq \frac{1}{100}$

From (iv), using (iii) & (v),

$k(x) \leq \frac{1}{320} + \frac{1}{2}k(x_0)$ (A)

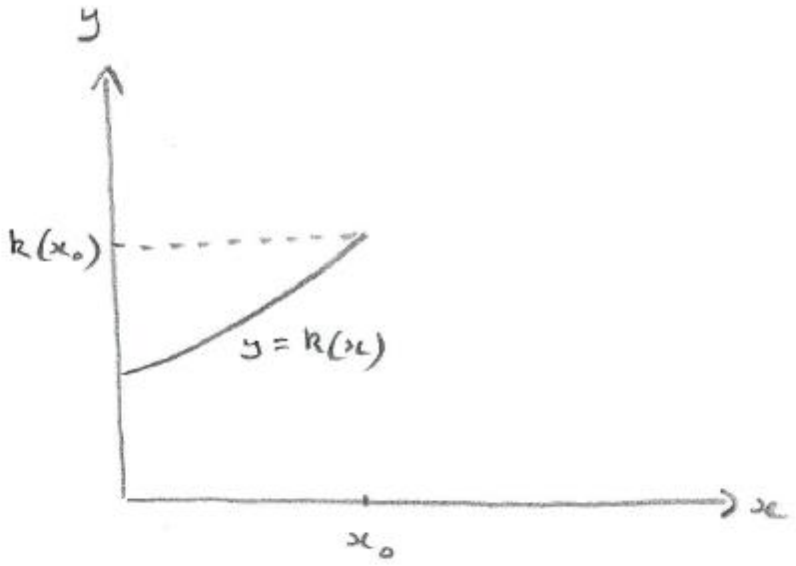
(since, from the working of (iii), $g(x) > f(x)$, so that

$g(x) - f(x) = |g(x) - f(x)| \leq \frac{1}{320}$)

Also, $k(x) \leq k(x_0)$ (B)

[At first sight, this doesn't look promising, as the inequalities in (A) & (B) seem to be in unfavourable directions:

$k(x) \leq \frac{1}{320} + \frac{1}{2}k(x_0) \Rightarrow k(x_0) \geq 2k(x) - \frac{1}{160}$, but this can't be usefully combined with (B).



However, if we consider a simple example of a graph of $k(x)$, with an upper limit of $k(x_0)$ [see diagram], and note that $k(x)$ can't be above

$\frac{1}{320} + \frac{1}{2}k(x_0)$, then we see that this doesn't work if $k(x_0)$ is very large relative to $\frac{1}{320}$, but that it can do if $k(x_0)$ is small enough relative to $\frac{1}{320}$

(in general, a useful device is to consider extreme situations)

So we need to be looking for an upper limit for $k(x_0)$.

From (A), $k(x) \leq \frac{1}{320} + \frac{1}{2}k(x_0)$ whenever $0 \leq x \leq \frac{1}{2}$

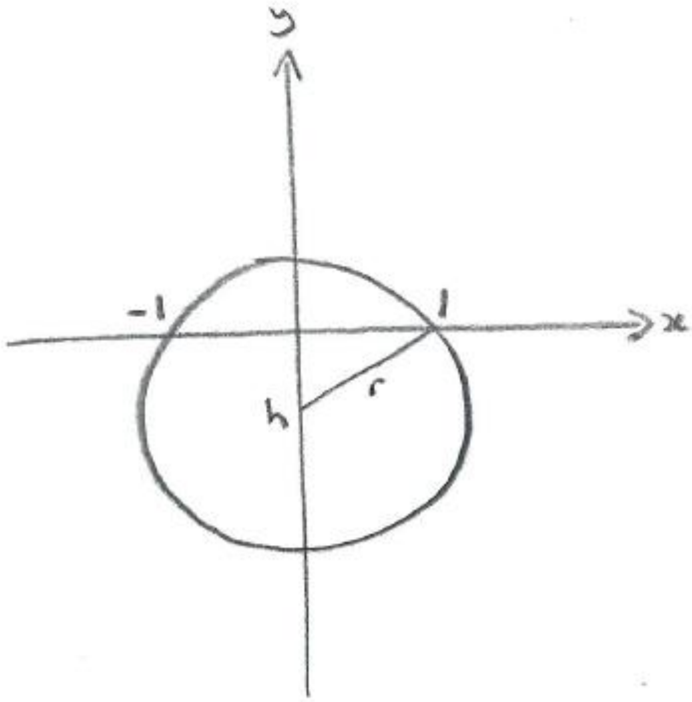
In particular, $k(x_0) \leq \frac{1}{320} + \frac{1}{2}k(x_0)$, so that $\frac{1}{2}k(x_0) \leq \frac{1}{320}$ and $k(x_0) \leq \frac{1}{160}$

Then $k(x) \leq k(x_0) \leq \frac{1}{160} < \frac{1}{100}$,

and $k(x) \leq \frac{1}{100}$, as required.

Q4 Sol'n / Notes

(i) [A diagram may reveal something that hasn't been considered.]



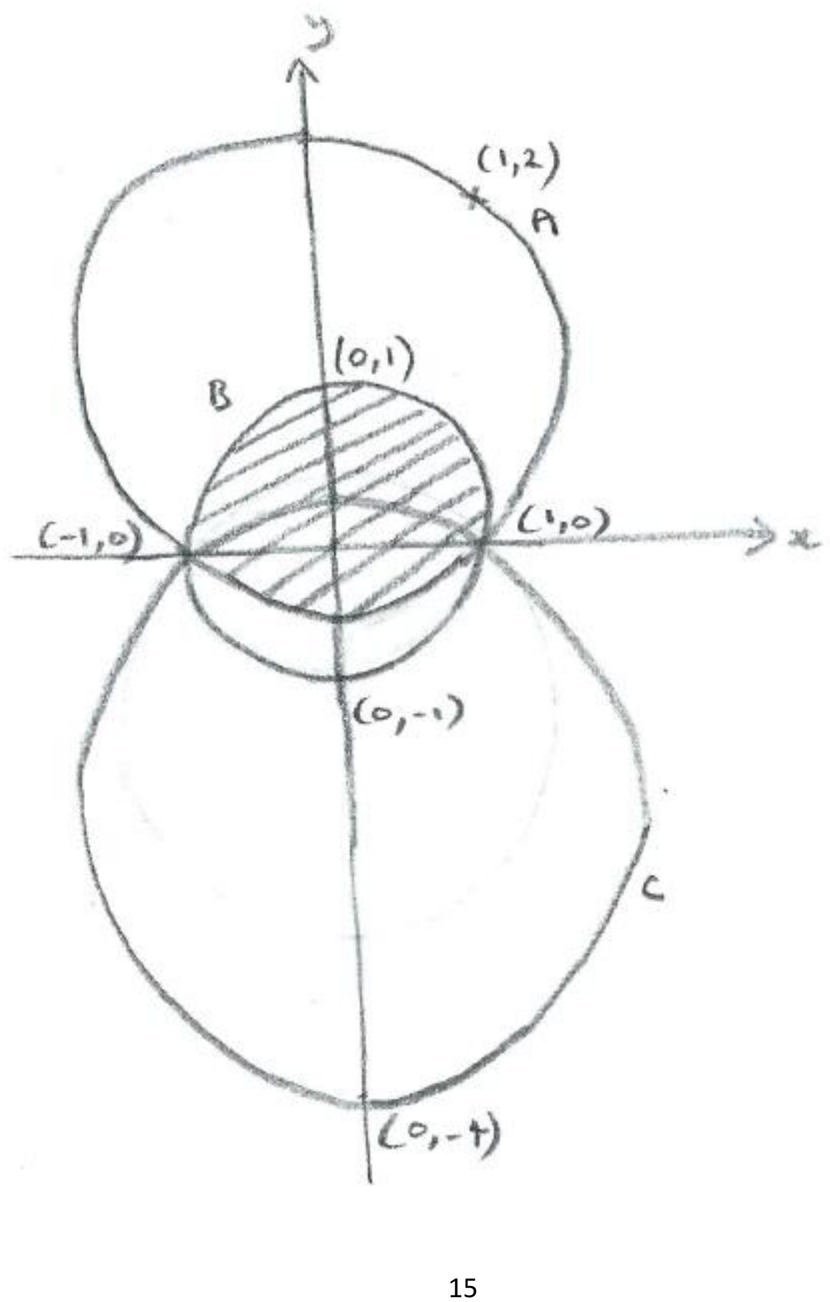
The diagram shows one possible configuration. We note that the centre of the circle will lie on the perpendicular bisector of the two points; ie the y-axis, and so $m = 0$.

The centre could also lie above the Origin, but in either case

$r = \sqrt{h^2 + 1}$

(ii) The centre of the circle lies on the y-axis and also on the perpendicular bisector of (for example) (1,0) & (x_0, y_0). Either $y_0 > 0$ or $y_0 < 0$, and from the diagram we can see that in each case the centre is uniquely defined as the intersection of these lines. The radius is then found from the centre and one of the 3 points, and so the circle is uniquely determined.

(iii) Referring to the diagram below, we only need to find the shaded region of B, A_1 say, as well as the area of A, A_2 say.



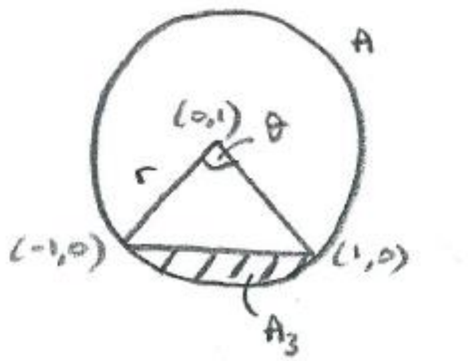
Then the lopsidedness of circle A is $1-\frac{A_1}{A_2}$

We note that the centre of circle A lies on the perpendicular bisector of the line joining (1,0) & (1,2), as well as being on the y-axis; so the centre is $(0,1)$, and the radius is seen to be $\sqrt{2}$

Hence $A_2=2\pi$

To find A_1, we need to determine the area of the segment of A, A_3 say, that lies below the y-axis. Referring to the diagram below,

$A_3=\frac{1}{2}r^2(\theta-sin\theta)$, where $r=\sqrt{2}$ and θ is seen to be $\frac{\pi}{2}$ (considering one of the right-angled triangles formed by bisecting the angle θ; the sides of which are $1, 1$ & $\sqrt{2}$)



So $A_3=\frac{1}{2}(2)\left(\frac{\pi}{2}-1\right)=\frac{\pi}{2}-1$

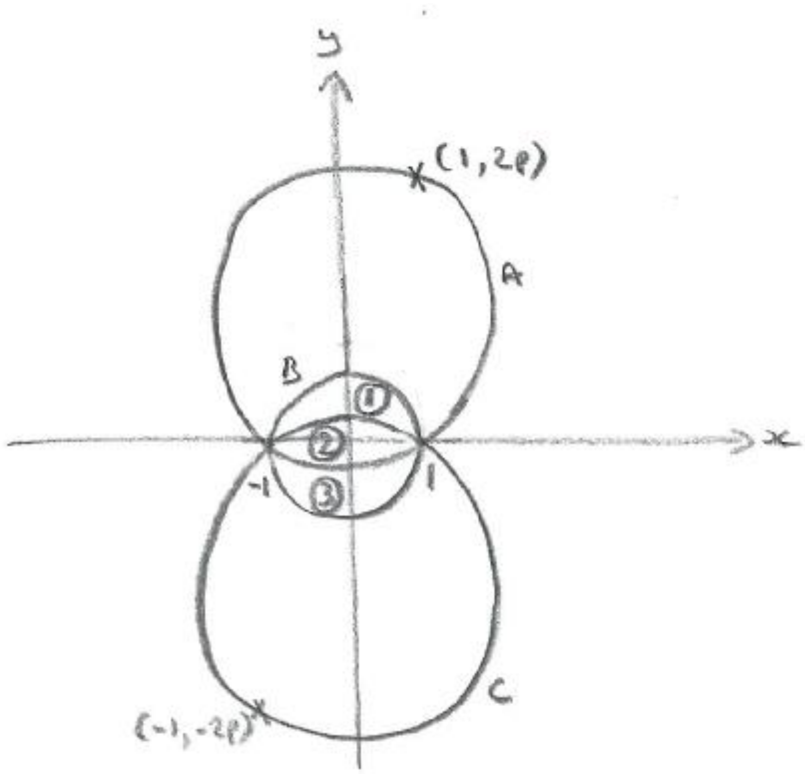
and $A_1=$ half of area of B $+\,A_3$

$=\frac{1}{2}\pi(1)^2+\frac{\pi}{2}-1=\pi-1$

Hence the lopsidedness of circle A is $1-\frac{A_1}{A_2}=1-\frac{\pi-1}{2\pi}$

$=\frac{1}{2}+\frac{1}{2\pi}$ or $\frac{\pi+1}{2\pi}$

(iv) The circumference of circle A will still be outside that of circle B, above the y-axis, as A has to pass through $(1, 2p)$. By symmetry, as circle C passes through $(-1, -2p)$, it is the reflection of A in the x-axis. See diagram below: regions 1 & 3 are equal.



The lopsidedness of B is the ratio of the largest region to the whole circle.

To get a feel for what is going on, we can consider two extreme cases: (a) big gap between A and B, and (b) small gap between A and B.

For case (a), region 1 (or 3) is the largest, and to minimise the ratio, we want region 1 (and 3) to equal region 2.

For case (b), region 2 is the largest, and to minimise the ratio, we want region 2 to equal region 1 (and 3) - again.

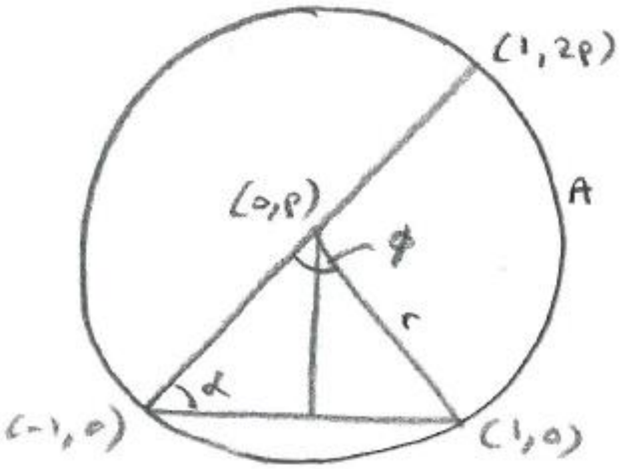
In other words, in all circumstances, we want the 3 regions to be equal.

This will be the case when region 2 has area of $\frac{1}{3}\times$ area of B; ie $\frac{\pi}{3}$

Similarly to part (ii), the area of region 2

$= 2\times\frac{1}{2}r^2(\phi - sin\phi)$ (see diagram below)

and hence $r^2(\phi - sin\phi) = \frac{\pi}{3}$



First of all, $r^2 = p^2 + 1$ (from the diagram),

so that $(p^2 + 1)\ (\phi - sin\phi) = \frac{\pi}{3}$ (*)

The presence of $\frac{\pi}{6}$ in the given result suggests that we should be looking at $\frac{\phi}{2}$, and we want $\frac{\phi}{2} = tan^{-1}(\frac{1}{p})$

From the diagram, $\tan\left(\frac{\phi}{2}\right) = \frac{1}{p}$, as required.

Also, $\frac{sin\phi}{2} = \frac{sin\alpha}{r} = \frac{(p/r)}{r} = \frac{p}{p^2+1}$

Thus $(*) \Rightarrow (p^2 + 1)tan^{-1}\left(\frac{1}{p}\right) - p = \frac{\pi}{6}$, as required.

Q5 Sol'n / Notes

(i) $s\left(p(0), m(0), m(m(0))\right) = s(1, -1, -2) = -2$

$s\left(p(0), m(0), p(p(0))\right) = s(1, -1, 2) = 2$

$s\left(m(0), p(0), m(p(0))\right) = s(-1, 1, 0) = 1$

Hence the given expression is $s(-2, 2, 1) = 2$ as required.

(ii) [It may be worth experimenting with substituting in 5 and 2 initially, but once an iterative relation becomes apparent, it is probably safer to revert to $a\ and\ b$.]

$f(a, m(b)) = f(a, b-1)$

$p(f(a, b-1)) = f(a, b-1) + 1$

$f(a, b) = s(b, p(a), f(a, b-1) + 1\ )$

$= s(b, a+1, f(a, b-1) + 1)$

If $b \leq 0$, $f(a, b) = a + 1$ (*)

If $b > 0, f(a, b) = f(a, b-1) + 1$ (**)

So $f(5, 2) = f(5, 1) + 1$

$= (f(5, 0) + 1) + 1$

$= f(5, 0) + 2$

$= (5 + 1) + 2 = 8$

(iii) With $b > 0, f(a,b) = f(a,b-1) + 1$, from (**) in (ii).

As $f(a,0) = a + 1$, from (*) in (ii),

we have an arithmetic sequence where $f(a,n) = (a+1) + n$;

ie $f(a,b) = (a+1) + b = a + b + 1$

(iv) We want $g(a,b) = a + b$ for $b \leq 0$

So $g(a,-2) = a - 2$

$g(a,-1) = a - 1$

$g(a,0) = a$

As we are to use $s(x,y,z)$, we need to have 2 cases: $x \leq 0$

and $x > 0$.

So, with $x = b$ (perhaps), we would like:

If $b \leq 0, g(a,b) = g(a,b+1) - 1$

This gives $g(a,-2) = g(a,-1) - 1$

$g(a,-1) = g(a,0) - 1$

$g(a,0) = g(a,1) - 1$

and we want $g(a,0) = a$, so we need $g(a,1) = a + 1$

For example, if $b > 0, g(a,b) = a + 1$ will do

Using $s(x,y,z)$, we can write this as:

$g(a,b) = s(b, g(a,b+1) - 1, a + 1)$

or $s(b, m[g(a,b+1)], p(a))$

or $s(b, m[g(a,p(b))], p(a))$